A Mathematical Error in the Lienard–Wiechert Retarded Potentials

J.P. Wesley

Abstract
The derivation of the Lienert–Wiechert retarded potential involves a mathematical error, so it is not a valid solution to the inhomogeneous wave equation, and it does not represent retarded action correctly. The correct retarded potential satisfying the inhomogeneous wave equation is presented, which agrees with the independent result derived from first principles based directly upon retarded action.

Key words: retarded potential, Lienard–Wiechert wrong, electrodynamics

1. INTRODUCTION

The Lienard[1]–Wiechert[2] retarded potentials were proposed for electrodynamics phenomena propagated in absolute space or in a luminiferous ether, as predicted by Maxwell theory.

The concept of retarded time is very important. If action is propagated with a finite velocity c, instead of instantaneously, then introducing time retardation into static or steady-state potential fields yields propagation of these fields. In particular, retarded potentials must obey the wave equation with the phase velocity c.

Unfortunately Lienard and Wiechert included a mathematical mistake in their analysis of time retardation, which makes their expressions for the retarded potentials wrong. These incorrect Lienard–Wiechert expressions for the retarded potentials continue to be reproduced in electrodynamics textbooks.[3] The correct mathematical expression for a retarded potential is presented below.

(The Lienard–Wiechert retarded potentials have been recently criticized[4,5] in terms of "special relativity." Since the Lienard–Wiechert potentials were proposed before special relativity, since special relativity is not valid,[6] and since this criticism does not include the mathematical error considered here, this criticism is not relevant to the present paper.)

2. THE PHYSICS IMPLIED BY THE INHOMOGENEOUS WAVE EQUATION

To agree with the concept of time retardation the Lienard–Wiechert retarded potentials must be solutions to the wave equation.[7] To illustrate the principles involved it is sufficient to consider the scalar solution Φ to the inhomogeneous scalar wave equation

$$\nabla^2\Phi - \frac{\partial^2\Phi}{\partial t^2 c^2} = -4\pi\rho \tag{1}$$

for a source function $\rho = \rho(\mathbf{r}, t)$, where the wave velocity $\mathbf{c}$ is a constant in the $\mathbf{r}$ space for ∇^2. The propagation properties of the wave are independent of the source function ρ. In particular, in regions where $\rho = 0$, in empty space, an elementary plane-wave solution to (1) is given by

$$\Phi = \sin\left[\frac{2\pi(x - ct)}{\lambda}\right], \tag{2}$$

where λ is the wavelength, a parameter independent of c.

Since the wave velocity in (1) cannot be taken as dependent upon the velocity of a source charge $\mathbf{v}'$, the Ritz or ballistic theory for light, where the wave velocity is assumed to be $\mathbf{c} + \mathbf{v}'$, cannot be represented by the inhomogeneous wave equation (1).

Since the wave velocity $\mathbf{c}$ is constant in the $\mathbf{r}$ space, it cannot be taken as a function of the velocity $\mathbf{v}$ of an arbitrarily moving observer in the $\mathbf{r}$ space. In the observer's rest-frame, (1) then requires mathematically that the observer see a one-way wave velocity $\mathbf{c}^*$ given by

$$\mathbf{c}^* = \mathbf{c} - \mathbf{v}, \tag{3}$$

where the signs are chosen for the source approaching the observer and the observer receding from the source. This conclusion (3) is in agreement with the experimental observations of the one-way velocity of light.[8–14]

It may be concluded that the inhomogeneous wave equation (1) can only represent a physical wave in a preferred frame of reference, the $\mathbf{r}$ frame, in which the wave velocity $\mathbf{c}$ is a constant, such as a sound wave in a still gas or liquid, an elastic wave in a stationary solid, or a light wave in a fixed luminiferous ether or absolute space.

3. THE PROPAGATION OF ACTION

The effect of one body at $\mathbf{r}'$ on another body at $\mathbf{r}$ separated by the distance $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ may be assumed to propagate with a velocity of action $\mathbf{u}$.

According to Mach,[15] action at a distance is supposed to be instantaneous, so the velocity of action is infinite:

$$\mathbf{u}(\text{Mach}) = \infty. \tag{4}$$

This theory cannot account for the fact that light and electromagnetic signals are propagated with a finite, and not an infinite, velocity. Nor does Mach's pure relativity theory explain all the phenomena that depend upon absolute space.[16]

According to the Ritz[17] or ballistic theory the velocity of action is supposed to be propagated with the velocity $\mathbf{c}$, the velocity of light, with respect to a source body moving with the velocity $\mathbf{v}'$; thus,

$$\mathbf{u}(\text{Ritz}) = \mathbf{c} + \mathbf{v}'. \tag{5}$$

This theory does not agree with the relevant observations.[18]

A theory involving a velocity of action that depends upon the velocity of the observer $\mathbf{v}$ (such as special relativity, where the one-way absolute velocity of action would have to equal $\mathbf{c} + \mathbf{v}$ in order for $\mathbf{c}^*$ to always equal $\mathbf{c}$, as postulated) violates causality, as the source would have to have prior knowledge of the velocity of the observer before a signal or the action could proceed from the source. A prior cause cannot depend upon an effect that has not as yet even occurred.

According to classical theory action and light propagate with the velocity $\mathbf{c}$ independent of the motion of the source; thus,

$$\mathbf{u}(\text{classical}) = \mathbf{c}, \tag{6}$$

where $\mathbf{c}$ is the one-way velocity of light with respect to the fixed luminiferous ether or absolute space, as defined by the preferred $\mathbf{r}$ space in (1). This velocity of action, (6), is confirmed by *all* of the observations of the one-way velocity of energy propagation of light.[8–14]

A signal proceeding from a source with the instantaneous position $\mathbf{r}'(t)$ with the velocity of action $\mathbf{c}$, independent of the velocity of the source, requires the time Δt to reach an observer at the position $\mathbf{r}(t + \Delta t)$, where

$$\Delta t = \frac{|\mathbf{r}(t+\Delta t) - \mathbf{r}'(t)|}{c} = \frac{R_r}{c}, \tag{7}$$

where R_r is the retarded distance. If $\mathbf{R} = \mathbf{r}(t) - \mathbf{r}'(t)$ is the initial instantaneous separation distance between source and observer and if the observer moves with the constant velocity $\mathbf{v}$, then at the time $t + \Delta t$ the retarded separation distance $\mathbf{R}_r$ becomes

$$\mathbf{R}_r = \mathbf{R} + \mathbf{v}\Delta t = \mathbf{R} + \frac{\mathbf{v}R_r}{c}. \tag{8}$$

It may be noted that the initial instantaneous relative distance $\mathbf{R}$ is a function of the time and the relative velocity between the source and the observer. During the time Δt after the action or signal has left the source and before it has arrived at the observer the action itself is independent of both the source and the observer, as may be envisioned by a flight of photons moving with the velocity $\mathbf{c}$ relative to absolute space.

Solving (8) for R_r yields

$$R_r = \frac{\left[(\mathbf{v}\cdot\mathbf{R}/c) + \sqrt{(\mathbf{v}\cdot\mathbf{R}/c)^2 + (1 - v^2/c^2)R}\right]}{1 - v^2/c^2}. \tag{9}$$

For the case of an observer moving directly away from the source with the velocity $v < c$, (9) gives

$$R_r = \frac{R}{1 - v/c}, \tag{10}$$

and the retardation Δt becomes

$$\Delta t = \frac{R}{c - v}. \tag{11}$$

(One may, of course, choose the initial time as $t - \Delta t$ and the final time as t without these results (8) through (11) being altered.)

For the retarded Coulomb potential the apparent distance to the charge as seen by the moving observer is given by the retarded distance R_r (9). Thus, from first principles the retarded Coulomb potential is simply

$$\Phi = \frac{q}{R_r}. \tag{12}$$

For the observer moving directly away from the charge, where R_r is given by (10), the retarded Coulomb potential becomes

$$\Phi = \frac{q(1 - v/c)}{R}. \tag{13}$$

4. THE CORRECT RETARDED SOLUTION TO THE INHOMOGENEOUS WAVE EQUATION

The solution to the inhomogeneous wave equation (1) can be represented as an integral over the source function $\rho(\mathbf{r}, t)$ by introducing the Green's function that satisfies[7]

$$\left(\nabla^2 - \frac{\partial^2}{\partial^2 t^2}\right) G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{14}$$

where r' and t' are *independent* variables for any arbitrary space-time position and r and t represent the space-time position of a point source. Solving (14) for G vanishing at r', t' at infinity yields

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}. \tag{15}$$

Then in the usual way the solution to (1) can be represented by the integral expression

$$\Phi(\mathbf{r},t) = \int d^3r' \int dt' \rho(\mathbf{r}',t') \frac{\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}, \tag{16}$$

where the integration over $\mathbf{r}'$ and t' is taken over all space and time, where $\mathbf{r}'$ is *independent* of t'. Performing the time integration yields the correct retarded potential

$$\Phi_r(\mathbf{r},t) = \int d^3r' \frac{\rho(\mathbf{r}',\tau)}{|\mathbf{r}-\mathbf{r}'|}, \tag{17}$$

where τ is the retarded time defined by

$$\tau = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}. \tag{18}$$

The space integration over $\mathbf{r}'$ in (17) is to be carried out regarding τ as a constant because $\mathbf{r}'$, being independent of t', must also be taken as independent of τ.

5. THE CORRECT RETARDED COULOMB POTENTIAL FOR A MOVING CHARGE

Substituting the source function for a moving point charge q, as given by

$$\rho(\mathbf{r}',t) = q\delta[\mathbf{r}'-\mathbf{r}(t)], \tag{19}$$

where $t = \tau$, into (17) the integration over $\mathbf{r}'$ may be performed immediately, treating τ as a constant, to give the correct retarded Coulomb potential for a moving point charge as

$$\Phi_r = \frac{q}{R_r}, \tag{20}$$

where R_r is given by (9). This solution to the wave equation (1) may also be seen to be correct because it agrees with the Φ_r derived independently from first principles, given by (12).

6. THE INCORRECT LIENARD–WIECHERT RETARDED SOLUTION

Despite the clear mathematical requirement that r' and t' be independent variables, which must be thus independent of each other, which are to be integrated over all space and time, and which are introduced merely to generate an integral expression for the solution to the inhomogeneous wave equation (1), Lienard and Wiechert argue incorrectly that the *independent* space variable $\mathbf{r}'$ is a *dependent* function of the time variable t'. Thus, the delta function in (16) is incorrectly interpreted to mean[3]

$$\delta\left(t-t'-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right) = \delta\left(t-t'-\frac{|\mathbf{r}-\mathbf{r}'(t')|}{c}\right) = \delta[t-t'(R)], \tag{21}$$

as though $\mathbf{r}'$ were to represent the *dependent* position of a source charge at $\mathbf{r}'$ at the instant t', instead of being merely any *independent* position $\mathbf{r}'$ anywhere at all in space. Moreover, it is thus implicitly assumed here that the action proceeds with the velocity $\mathbf{c}$ with respect to a moving point source charge, or with the velocity $\mathbf{c} + \mathbf{v}'$, which is the Ritz or ballistic theory, which is empirically wrong. For the integration over t' the falsely interpreted delta function, given by (21), can be replaced by

$$\begin{aligned}\delta\left(t'-t-\frac{|\mathbf{r}-\mathbf{r}'(t')|}{c}\right) &= \frac{\delta(t'-t)}{d(t'-t-|\mathbf{r}-\mathbf{r}'(t')|/c)/dt'} \\ &= \frac{\delta(t-t')}{1-\mathbf{R}\cdot\mathbf{v}'/(cR)},\end{aligned} \tag{22}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'(t')$ and $\mathbf{v}' = d\mathbf{r}'/dt'$. Since $\mathbf{r}'$ must be independent of t', no $\mathbf{v}' = d\mathbf{r}'/dt'$ can be logically defined! Substituting the unjustified replacement (22) into (16) and performing the integration over t' yields the incorrect Lienard–Wiechert retarded potential Φ_{rLW} as

$$\Phi_{rLW} = \int d^3r' \frac{\rho(\mathbf{r}',\tau)}{R - \mathbf{R}\cdot\mathbf{v}'/c}, \tag{23}$$

where $\mathbf{R}$ and $\mathbf{v}'$ are to be evaluated at the retarded time (18).

For a moving point charge the incorrect Lienard–Wiechert potential is given by substituting the source function (19) for $t = \tau$ into (23) and integrating, yielding

$$\Phi_{rLW} = \frac{q}{R - \mathbf{R}\cdot\mathbf{v}'/c}, \tag{24}$$

where $\mathbf{R}$ and $\mathbf{v}'$ are to be evaluated for the retarded time (18). For the case where an observer moves with the velocity v directly away from the source charge collinearly moving with the velocity v', (24), using (10), gives

$$\Phi_{rLW} = \frac{q(1 - v/c)}{R(1 - v'/c)}. \tag{25}$$

This Lienard–Wiechert result (24) or (25) does not agree with the correct retarded Coulomb potential (12) or (13) derived from the inhomogeneous wave equation (1) and from first principles. Thus, the Lienard–Wiechert potential is not only not a proper solution to the inhomogeneous wave equation (1), it also violates first principles. It may be noted from (24) or (25) that the Lienard–Wiechert. retarded potential implies a velocity of action that is not c but depends upon the velocity of the source, contrary to observations.

Received 19 January 2000.

Résumé

La dérivation du potentiel retardé de Lienert–Wiechert comprend une erreur mathématique. Par conséquent elle n'est pas une solution valide pour l'équation d'onde inhomogène et elle ne représente pas l'action retardée correctement. La bonne valeur du potentiel retardé pouvant satisfaire l'équation d'onde inhomogène présentée est d'accord avec le résultat indépendant dérivé des premiers principes basés directement sur l'action retardée.

References

1. A. Lienard, Éclairage Électrique **16**, 5, 53, 106 (1989).
2. E. Wiechert, Arch. Neerlandes **5**, 549 (1900).
3. Any graduate textbook in electrodynamics, such as J.D. Jackson, *Classical Electrodynamics* (John Wiley and Sons, New York, 1965), pp. 464–468.
4. C.K. Whitney, Phys. Essays **1**, 15 (1988); **2**, 47 (1989); Hadronic J. **11**, 257 (1988).
5. D.P. Allen, Phys. Essays **6**, 133 (1992).
6. G. Galeczki and P. Marquardt, *Requiem für die Spezielle Relativität* (Haag and Herchen, Frankfurt am Main, Germany, 1997).
7. J.D. Jackson, *Classical Electrodynamics* (John Wiley and Sons, New York, 1965), pp. 183–186.
8. O. Roemer, Philos. Trans. **12**, 893 (1677).
9. J. Bradley, Philos. Trans. R. Soc. London **35**, 406 (1728).
10. G. Sagnac, Comptes-Rendus **157**, 708 (1913).
11. A.A. Michelson and H.G. Gale, Astrophys. J. **61**, 137 (1925).
12. E.K. Conklin, Nature **222**, 971 (1969).
13. S. Marinov, Czech. J. Phys. B **24**, 965 (1974); Gen. Relativ. Gravit. **12**, 57 (1980); Spec. Sci. Tech. **3**, 57 (1980).
14. C.M. Hill, Phys. Essays **3**, 429 (1990).
15. E. Mach, *Die Mechanik in Threr Entwicklung*, reprint of original 1933 edition (Wissenschaftliche Buchgesellschaft, Darmstadt, Germany, 1991).
16. J.P. Wesley, in *Open Questions in Relativistic Physics*, edited by Franco Selleri (Apeiron, Montreal, 1998), pp. 255–261.
17. W. Ritz, Ann. Chim. Phys. **13**, 145 (1908).
18. J.P. Wesley, *Advanced Fundamental Physics* (Benjamin Wesley, Blumberg, Germany, 1991), pp. 81–85.

J.P. Wesley
Weiherdammstrasse 24
78176 Blumberg, Germany