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Ampere's Original Force Law Compared with the Moyssides-Pappas Results

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The force on Ampere's bridge with straight ends (a current carrying π-shaped wire frame) due to the remainder of the circuit is derived correctly from Ampere's original differential force law for the first time without any amending factors. The theory is in reasonable agreement with the variation of the force as a function of the diameter of the wire as measured by Moyssides and Pappas[1] (J. Appl. Phys. **59** (1986)). The force on Ampere's bridge with bent ends is also derived. These results differ from measurments by 20-30%. However, if a small systematic experimental error is postulated, there is agreement with the theory.

I. INTRODUCTION

Ampere's original differential force law[2] states that the force $d^2\mathbf{F}$ on a current element $I_2 d\mathbf{s}_2$ due to a current element $I_1 d\mathbf{s}_1$ which are separated by the distance $r = \mathbf{r}_2 - \mathbf{r}_1$ is given by

$$d^2\mathbf{F} = I_2 I_1 \mathbf{r}\left(-2(d\mathbf{s}_2 \cdot d\mathbf{s}_1)/r^3 + 3(d\mathbf{s}_2 \cdot \mathbf{r})(d\mathbf{s}_1 \cdot \mathbf{r})/r^5\right), \qquad (1)$$

in c.g.s. units where I_2 and I_1 are in abamperes. This law has been rigorously quantitatively confirmed where the force between a closed current loop and a current element (or a moving charge) or between two closed current loops is involved. For these integral cases Ampere's law forms the basis of traditional magnetostatics. However, since the fundamental force law between two moving point charges, as prescribed by Eq. (1) where $I d\mathbf{s} = q\mathbf{v}$, charge times velocity, does not in general involve the effect of a closed current loop; it is most important to also test Ampere's law quantitatively when no closed current loops are involved. Only in this way is it possible to test the correctness of the *differential* form of Ampere's law.

As recognized by Ampere himself, a crucial experiment involves obtaining quantitatively the force on Ampere's bridge[2], a movable current-carrying π-shaped wire frame with legs making electrical contact in mercury cups, due to the remainder of the circuit. Although the current flows around a closed loop; the bridge, being *mechanically* independent from the rest of the loop, must experience a force due to the current in the bridge only interacting with the current in the remainder of the loop. Neither the bridge all-by-itself nor the remainder of the loop all-by-itself form closed current loops; thus, the traditional theory is not applicable. Ampere[2], Hering[3], Cleveland[4], Pappas[5] (in a paper prior to Ref. (1)), and Graneau[6] have shown that the bridge is repelled from the remainder of the circuit as would be expected from Ampere's law (1); but they obtained no accurate quantitative results. The difficulties have been both experimental and theoretical. A valid expression for the force on Ampere's bridge derived from Ampere's law, that could be compared with experiment, was not available; and adequate experimental data were not available. But now Moyssides and Pappas have obtained, for the first time in the 160 year history of the problem, adequate quantitative measurements of the force on Ampere's bridge that can be used to compare with theory. The requisite correct theory to compare with their experimental results is derived here for the first time. A quantitative test of Ampere's *differential* law is now possible. Moyssides and Pappas measured the force on Ampere's bridge with straight ends and with bent ends. Both situations are treated theoretically below.

The theoretical difficulty in the past has arisen from the fact that Ampere's law, as given by Eq. (1), yields an infinite force when two linear current elements are brought together, the force varying as the inverse square of the separation distance. As in the electrostatic case where similar infinities predicted by Coulomb's law for point charges, are eliminated by turning to volume charge densities, the infinities predicted by the linear form of Ampere's law (1) can be eliminated by turning to volume current densities $\mathbf{J}_2$ and $\mathbf{J}_1$; thus,

$$d^6\mathbf{F}/d^3r_2d^3r_1 = \mathbf{r}\left[-2\mathbf{J}_2\cdot\mathbf{J}_1/r^3 + 3(\mathbf{J}_2\cdot\mathbf{r})(\mathbf{J}_1\cdot\mathbf{r})/r^5\right]. \qquad (2)$$

It may be readily proved that integrating Eq. (2) over continuous finite current density distributions can yield no infinities, in agreement with laboratory observations.

Investigators in the past have attempted to calculate the force on Ampere's bridge, where the force on neighboring current elememts with zero separation distance occur, by incorrectly integrating Eq. (1) for current filaments of vanishing cross section. To remove the inevitable infinities that must arise amending factors have been introduced. Such theoretical results cannot be compared with experiments; as the largest effect is produced by the correction factors themselves. Cleveland[4] terminated his integration by an explicit arbitrary finite separation distance. Robertson[7] claimed incorrectly that his correction factor corresponded to the diameter of the wire used. Both Cleveland and Robertson have additional errors in sign. Graneau[6] chooses to introduce an arbitrary shortest size interval as a factor in his computer summations of Eq. (1). No arbitrary factors are involved here; as the correct Eq. (2) is used where no singularities can arise.

II. FORCE ON AMPERE'S BRIDGE WITH STRAIGHT ENDS

Figure 1 shows the circuit for the Ampere bridge experiment with the choice of coordinates, labels, and geometry. Three dimensional laminar geometry is assumed; the third, or z, direction is not indicated in Fig. 1. The fixed portion of the circuit containing the battery consists of portions 1 (between 0,0 and L,0), 2 (between L,0 and L,N), and 10 (between 0,0 and 0,N). The bridge consists of portions 5 (between L,N and L,M), 6 (between 0,M and L,M), and 7 (between 0,N and 0,M). Electrical contact with the bridge is made through cups containing mercury at 0,N and L,N.

The situation of interest is for the conductor width, w, and the laminar thickness, t , small compared with the other dimensions of the circuit. For the force between portions of the circuit not in contact (1&6, 1&5, 1&7, 2&6, 2&7, 10&6, and 10&5) it is possible to replace the volume integrations indicated by Eq. (2) by line integrations, indicated by Ampere's original formula (1), by integrating transverse to the direction of current flow and using the mean value theorem for integrals. The force between portions of the circuit in contact with each other (2&5 and 10&7) cannot be calculated using such an approximation. In this case the approximation, Eq. (1), would yield artificial singularities. For the force between portions in contact volume integrations are unavoidable.

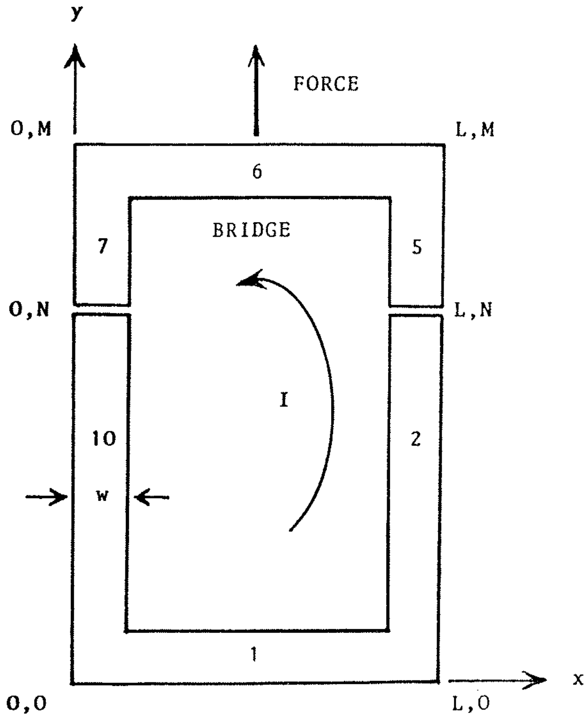

FIG. 1. Diagram of the experiment for the force on Ampere's bridge with straight ends indicating coordinates, labeling, and geometry.

The force F' on the bridge, which is in the y direction, due to portions not in contact (1&6, 1&5, 1&7, 2&6, 2&7, 10&6, and 10&5) may be obtained by a straightforward integration using Eq. (1), where Jwt = I, yielding in closed mathematical form

$$F'/2I^2 = \sqrt{1 + L^2/M^2} - ln\left((M - N)/M\right) - ln(N/L) - ln\left[1 + \sqrt{1 + L^2/M^2}\right]. \quad (3)$$

The force F'' due to the portions in contact (2&5 and 10&7) is given from Eq. (2) by the integral

$$F'' = 2J^2 \int_0^t dz_2 \int_N^M dy_2 \int_0^w dx_2 \int_0^t dz_1 \int_0^N dy_1 \int_0^w dx_1 \left[-2Y/r^3 + 3Y^3/r^5 \right], \qquad (4)$$

where

$$Y = y_2 - y_1 . \qquad (5)$$

Integrating with respect to y_1 and y_2, Eq. (4) yields

$$F'' = 2J^2 \int_0^t dz_2 \int_0^w dx_2 \int_0^t dz_1 \int_0^w dx_1 \left\{ \frac{M}{\sqrt{M^2 + R^2}} - \frac{M - N}{\sqrt{(M - N)^2 + R^2}} - \frac{N}{\sqrt{N^2 + R^2}} + ln\left[\frac{M - N + \sqrt{(M - N)^2 + R^2}}{M + \sqrt{M^2 + R^2}}\right] + ln\left[\frac{N + \sqrt{N^2 + R^2}}{R}\right]\right\}, \qquad (6)$$

where

$$R^2 = (x_2 - x_1)^2 + (z_2 - z_1)^2 . \qquad (7)$$

Since w and t are small, most of the remaining integrals in Eq. (6) can be immediately evaluated by the mean value theorem for integrals for $w \to 0$ and $t \to 0$, or for $R \to 0$, where $Jwt = I$. Thus, Eq. (6) yields

$$F'' = 2I^2\left\{ -1 - ln[M/(M - N)] + ln2N \right\} + F''_s, \qquad (8)$$

where the so-called singularity term F''_s is

$$F''_s = -2J^2 \int_0^t dz_2 \int_0^w dx_2 \int_0^t dz_1 \int_0^w dx_1 \; lnR. \qquad (9)$$

The remaining four integrations indicated in Eq. (9) can be readily carried out in closed mathematical form using elementary functions, yielding without any approximations

$$F''_s = (J^2/3)\left[25w^2t^2/2 - w^4 lnw - t^4 lnt - (6w^2t^2 - t^4 - w^4) ln\sqrt{w^2 + t^2} - 4w^3t \tan^{-1}(t/w) - 4wt^3 \tan^{-1}(w/t)\right]. \qquad (10)$$

For the case of interest here the conductor width w may be taken equal to the laminar thickness t, or w = t; and F''_s, as given by Eq. (10), becomes

$$F_s'' = 2I^2\left[25/12 - \pi/3 - (1/3)\ln 2 - \ln w\right], \tag{11}$$

where again no approximations are involved.

Combining results, F' + F'', as given by Eqs. (11), (8), and (3), the net force on Ampere's bridge with straight ends F is given by

$$F/2I^2 = 13/12 - \pi/3 - (2/3)\ln 2 + \sqrt{1 + L^2/M^2} - \ln\left(1 + \sqrt{1 + L^2/M^2}\right) + \ln(L/w). \tag{12}$$

This theoretical result (12) is compared with the experimental results of Moyssides and Pappas below.

III. FORCE ON AMPERE'S BRIDGE WITH BENT ENDS

Moyssides and Pappas have measured the force on a bridge with bent ends as indicated in Fig. 2. The force on the bridge, assuming laminar geometry, can be computed in closed form from Eq. (1) precisely as used to derive the result (3). No singularity term involving $\ln w$ occurs in this case. The only place where current elements come together are between portions 3&4 and 9&8. No net force on the bridge is involved for the forces between portions 3&4 and 9&8. In the x direction they are equal and oppositely directed, and in the y direction no force is generated; as may be seen from symmetry or from Ampere's law. In this way Moyssides and Pappas were able to eliminate the variation with w experimentally.

Carrying out the integrations indicated by Eq. (1) and Fig. 2 yields

$$\begin{aligned} F/2I^2 = {} & \ln\left((L - P)/P\right) + \ln\left(Q/(Q - P)\right) + \sqrt{1 + Q^2/N^2} - \sqrt{1 + Q^2/M^2} \\ & + \sqrt{1 + Q^2/(M - N)^2} - \sqrt{1 + (L - Q)^2/N^2} + \sqrt{1 + (L - Q)^2/M^2} \\ & - \sqrt{1 + (L - Q)^2/(M - N)^2} - \sqrt{1 + P^2/N^2} - \sqrt{1 + (Q - P)^2/(M - N)^2} \\ & + \sqrt{1 + (L - Q - P)^2/(M - N)^2} + \sqrt{1 + (L - P)^2/N^2} - \ln\left((L - Q)/(L - Q - P)\right) \\ & + \ln\left[\frac{1 + \sqrt{1 + (L - Q)^2/N^2}}{1 + \sqrt{1 + Q^2/N^2}}\right] - \ln\left[\frac{1 + \sqrt{1 + (L - Q)^2/M^2}}{1 + \sqrt{1 + Q^2/M^2}}\right] \\ & - \ln\left[\frac{1 + \sqrt{1 + (L - P)^2/N^2}}{1 + \sqrt{1 + P^2/N^2}}\right] - \ln\left[\frac{1 + \sqrt{1 + Q^2/(M - N)^2}}{1 + \sqrt{1 + (Q - P)^2/(M - N)^2}}\right] \\ & + \ln\left[\frac{1 + \sqrt{1 + (L - Q)^2/(M - N)^2}}{1 + \sqrt{1 + (L - Q - P)^2/(M - N)^2}}\right]. \end{aligned} \tag{13}$$

Although this result (9) is a bit lengthy with 18 terms; 5 parameters are involved, and the geometry of the experimental setup, as shown in

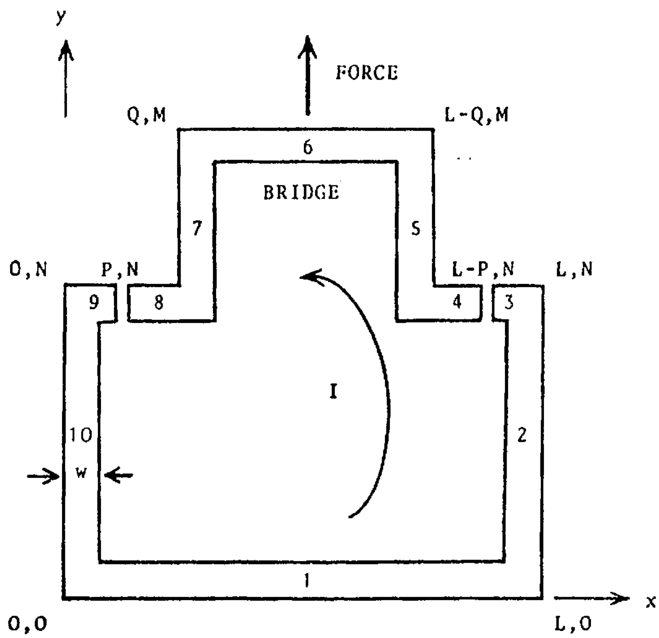

FIG. 2. Diagram of the experiment for the force on Ampere's bridge with bent ends indicating coordinates, labeling, and geometry.

Fig. 2, is rather complicated. However, numerical results from Eq. (13) may be readily obtained to compare with experiment.

IV. COMPARISON WITH EXPERIMENT FOR AMPERE'S BRIDGE WITH STRAIGHT ENDS

The geometry assumed for the theory assumes a rectangular cross section for the wire used; whereas Moyssides and Pappas used wires of circular cross section. To an adequate approximation the cross-sectional areas may be equated; or

$$w = \sqrt{\pi}\, d/2, \tag{14}$$

where d is the circular wire diameter. Moyssides and Pappas used for this case L = 48 cm and M = 120 cm. They used units of gram weight for the force F; so Eq. (12) must be divided by the acceleration of gravity 980.0 cm/sec². They used units of ampere for the current instead of abampere; so Eq.(12) must be divided by 100. Using Eq. (14) and the

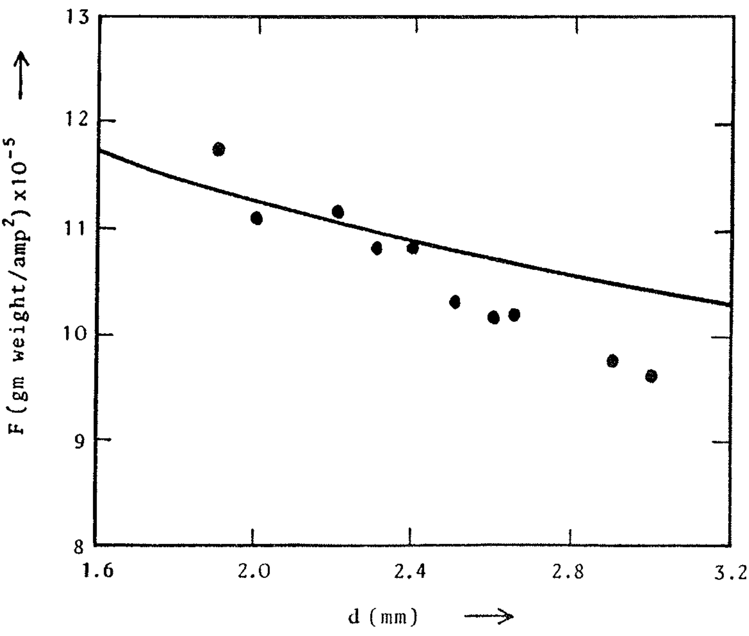

FIG. 3. Theoretical result (solid curve) as given by Eq. (15) (from Eq. (12)) compared with the experimental points of Moyssides and Pappas[1].

above facts, Eq. (12) yields the theoretical formula

$$F/I^2 = 12.683 - 2.041\, \ln d, \qquad (15)$$

where F is the force in gram weight units, I is the current in amperes, and d is the wire diameter in millimeters. This theoretical result (15) is compared with the experimental results of Moyssides and Pappas[1] (as presented in their Fig. 3) in Fig. 3.

The agreement is satisfactory considering the error of about ± 2 percent in the experimental observations. The experimental points drop off faster with d than the theoretical curve. This discrepancy may be due to a systematic experimental error as discussed below in Section VI.

V. COMPARISON WITH EXPERIMENTS FOR AMPERE'S BRIDGE WITH BENT ENDS

For the case of ends bent 1 cm, Q - P = 1 cm, L = 52 cm, P = 1 cm, M = 120 cm, and N = 43 cm Moyssides and Pappas report a force on Ampere's bridge per current² of $7.04 \pm 0.14 \times 10^{-5}$ gm weight/amp², where the error has been estimated from their Fig. 11. Substituting the dimensions reported by Moyssides and Pappas into Eq. (13) yields the theoretical prediction of 9.500×10^{-5} gm weight/amp². Similarly for the case of cm bent ends, where Q - P = 2 cm, L = 54 cm, P = 1 cm, M = 120 cm, and N = 43 cm, Moyssides and Pappas report a force per current² of $6.06 \pm 0.12 \times 10^{-5}$ gm weight/amp². The theoretical prediction in this case from Eq. (13) is 9.019×10^{-5} gm weight/amp². Results are summarized in Table I.

Table I. Force per current² on Ampere's bridge with bent ends in gm weight/amp² x 10^{-5}.

length of bent ends	experiment [1]	theory (Eq. (13))
1 cm	7.04 ± 0.14	9.500
2 cm	6.06 ± 0.12	9.019

The theoretical predictions are higher than the experimental observations. The discrepancy, which is not great considering all of the experimental and computational difficulties, may be due to a systematic experimental error as explained in the following Section.

VI. DISCUSSION AND CONCLUSIONS

Considering the well established success of the original Ampere law (1) or (2) in accurately predicting a huge mass of experimental data where the force on (or due to) a closed current loop is involved, a reason must be sought for the discrepancy mentioned above between the Ampere theory and the experiments of Moyssides and Pappas. It is quite significant that, independent of the various circumstances involved, the reported forces per current² are less than the theoretical values in a very regular way. In particular, for all 11 observations where the wire diameter was varied for the force on Ampere's bridge with straight ends and the two observations of the force on Ampere's bridge with ends bent 1 cm and 2 cm, the discrepancy $\Delta = \left[(F/I^2)_{theory} - (F/I^2)_{experiment}\right]$ in gm weight/amp² x 10^{-5} is given quite accurately to within the experimental errors by

$$\Delta = 16.5 - 1.50(F/I^2)_{theory} \tag{16}$$

Since this result (16) would seem to be independent of any of the independent variables, the shape and dimensions of the circuit and the diameter of the wire; there would seem to be systematic errors involved in the determination of the force F and or else of the current I.

Since all measurements were made with the same large (5 cm inner diameter) mercury cups; the systematic discrepancy in measuring F/I^2 might be a peculiar feature of the large mercury cups used. It is difficult to envision a mechanism that might give rise to such an effect. Never-the-less, varying the geometry and size of the cups should resolve the question.

In order to test Ampere's differential force law (1) or (2) in other particulars the force on Ampere's bridge should be measured as a function of the width of the bridge L with other parameters held fixed, the height of the bridge M - N with other parameters held fixed, and the height of the fixed curcuit N with other parameters held fixed. The need for such observations is indicated, for example, by the fact that for the particular dimensions chosen by Moyssides and Pappas for their Ampere bridge with bent ends about 96 percent of the force is predicted merely by the first two terms on the right of Eq. (13), which would hardly be universally true.

It is concluded that the experimental determinations of the force on Ampere's bridge by Moyssides and Pappas[1] confirm Ampere's original force law quantitatively in its differential form. The discrepancies between observations and theory may be regarded as small considering the experimental and theoretical difficulties. The regularity of these discrepancies indicate that they are probably systematic errors in the determination of the force F and or else the current I.

[1]P. G. Moyssides and Pappas, J. Appl. Phys. 59, 19 (1986).
[2]A. M. Ampere, Mem. Acad. R. Sci. (Paris) (1823); and Memoires sur l'Electrodynamique, 1, 25 (Gauthier Villars, Paris, 1882).
[3]C. Hering, Trans. Am. Inst. Electr. Eng. 43, 311 (1923).
[4]F. F. Cleveland, Philos. Mag. Suppl. 21, 416 (1936).
[5]P. T. Pappas, Nuovo Cimento B 78, 189 (1983).
[6]P. Graneau, J. Appl. Phys. 53, 6648 (1982); Phys. Lett. 107A, 235 (1985); IEEE Trans. Magn. MAG-20, 444 (1984); Nuovo Cimento, 78B, 231 (1983); and P. Graneau and P. N. Graneau, Appl. Phys. Lett. 46, 468 (1985).
[7]I. A. Robertson, Philos. Mag. 36, 32 (1945).