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Causal Quantum Mechanics with Phase and Particle Velocities Equal.

J. P. WESLEY

University of Missouri at Rolla - Rolla, Mo.

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Summary. — A relativistic theory of causal quantum mechanics for a scalar particle is proposed in which the phase velocity is equal to the particle velocity. A particle is viewed as a classical point particle that moves along a wave normal attached to a surface of constant phase. A prescription is derived for the particle trajectories in a standing wave which gives rise to periodic motion in the bound-particle case. The old quantum theory of Bohr and Sommerfeld is obtained in the geometrical-optics approximation. The observed particle density is the wave intensity (in agreement with $\Psi\Psi^*$ of the traditional theory). The problems of a free particle, a particle reflected from a mirror, and the simple harmonic oscillator are considered.

1. – Introduction.

From diffraction experiments the wavelength, λ, to be associated with a moving particle is the de Broglie wavelength $\lambda = 2\pi\hbar/|\boldsymbol{p}|$ where $\boldsymbol{p}$ is the momentum of the particle. Since the velocity of energy propagation and momentum transfer is the velocity of the particle $\boldsymbol{v}$, a plane wave for a free particle may be postulated in the form

$$\Psi(\boldsymbol{r}, t) = \sin[\boldsymbol{p}\cdot(\boldsymbol{r} - \boldsymbol{v}t)/\hbar], \tag{1}$$

where $\boldsymbol{p}$ and $\boldsymbol{v}$ are constants of the motion and $\boldsymbol{r}$ and t are the position and time. As an empirical equation that describes the essential laboratory results eq. (1) is certainly correct.

SIDDIQI [1] has presented an interesting causal theory in which he has also

(1) M. A. SIDDIQI: *Pakistan. Journ. Sci. Industr. Res.*, 6, 28 (1963).

chosen the phase and particle velocities equal. He unfortunatly interprets Ψ as a physical pilot wave. He does not consider the bound-particle case.

Pure real representations have been chosen here (there being no a priori reason for excluding such representations), in order to avoid the irreplaceable $i = \sqrt{-1}$ that occurs in the traditional theory [2,3].

The scalar Ψ field defined by eq. (1) is not Lorentz-invariant for arbitrary choices of the position, $\boldsymbol{r}$, and the time, t. Nevertheless, for values of $\boldsymbol{r}$ and t restricted to lie along the trajectory, $\boldsymbol{r} = \boldsymbol{r}(t)$, the phase in eq. (1), being always identically constant, does remain Lorentz-invariant. The Ψ function, regarded as a generating function, need only generate proper relativistic particle trajectories; the entire Ψ field need not have physical meaning; and Lorentz covariance for the entire field is unnecessary [1].

It may be seen that while the space part of the phase of the postulated plane wave for a free particle, eq. (1), is the same as the traditional theory, the time part differs. The quantity $\boldsymbol{p}\cdot\boldsymbol{v}/\hbar$ has replaced the traditional Planck-Einstein frequency, $\varepsilon/\hbar$, where ε is the total relativistic energy. It is important to note that for photons, where $\boldsymbol{p} = \varepsilon\boldsymbol{c}/c^2$, the theory proposed here preserves the Planck-Einstein frequency condition in agreement with observation [1]. Because the laboratory evidence for quantum-mechanical behavior does not involve explicit measurements of time, the validity of the time variation proposed here can only be tested theoretically (cf. [4]).

The traditional de Broglie wave [5] for a free particle,

$$\Psi = \sin\left[(\boldsymbol{p}\cdot\boldsymbol{r} - \varepsilon t)/\hbar\right], \tag{2}$$

suffers from a number of inherent difficulties. The relativistic translation of a system oscillating with the frequency $mc^2/\hbar$, where m is the mass of the system, yelds a standing wave and not the propagating de Broglie wave, eq. (2), as is sometimes claimed [6]. The phase velocity c^2/v is fictitious since it cannot be observed. If a group wave is introduced further difficulties arise. A group wave cannot exist in empty space [7]. The group wavelength appears to imply a particle that is not only too large but is also variable in size.

For the traditional nonrelativistic theory of a free particle the total relativistic energy ε in eq. (2) is replaced by the total classical energy E, the rest-mass energy being dropped. Unfortunately, the nonrelativistic phase velocity

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[2] A. Landé: *From Dualism to Unity in Quantum Physics* (Cambridge, 1960).
[3] J. P. Wesley: *Phys. Rev.*, **122**, 1932 (1961).
[4] Y. Aharonov and D. Bohm: *Phys. Rev.*, **122**, 1649 (1961).
[5] L. de Broglie: *Ann. de Phys.*, **3**, 22 (1925).
[6] P. G. Bergmann: *Introduction to the Theory of Relativity* (New York, 1942), p. 143.
[7] J. A. Stratton: *Electromagnetic Theory* (New York, 1941), p. 330.

$v/2$ cannot be obtained from the relativistic phase velocity c^2/v as $v \to 0$; and the phase is no longer either Lorentz or Galilean invariant. The transition from relativistic velocities to nonrelativistic velocities occurs smoothly in the theory proposed here. The angular frequency $\boldsymbol{p}\cdot\boldsymbol{v}/\hbar$ for a free particle, eq. (1), becomes $2E/\hbar$ in the non relativistic case — a factor of 2 greater than the traditional value.

2. – Waves and classical particle trajectories.

Formally unquantized classical particle motion may be generated from wave functions which have no direct physical reality. This Section shows how such wave functions may be found for both traveling waves and standing waves and how the classical trajectories may be obtained from such functions.

2·1. *Wave equation.* – A relativistically invariant scalar wave equation satisfied by eq. (1) for a free particle is

$$(c^2\boldsymbol{p}\cdot\nabla + \varepsilon\partial/\partial t)\,\Psi = 0\,. \tag{3}$$

For motion in either the positive or negative direction along the wave normal the postulated equation is

$$(c^2\boldsymbol{p}\cdot\nabla + \varepsilon\partial/\partial t)(c^2\boldsymbol{p}\cdot\nabla - \varepsilon\partial/\partial t)\,\Psi = 0\,. \tag{4}$$

This eq. (4), while no longer relativistically covariant, nevertheless, admits the two covariant possibilities. For $\boldsymbol{p}$ and ε constant and $\boldsymbol{p}$ parallel to the ∇ operator (as will be assumed to always be the case (see Sect. **2**·2)) eq. (4) becomes

$$(c^4p^2\nabla^2 - \varepsilon^2\partial^2/\partial t^2)\,\Psi = 0\,. \tag{5}$$

For the case of a particle moving under the action of a potential, V, or under the action of boundaries the scalar wave eq. (5) is generalized by replacing ε by $(\varepsilon - V)$; thus,

$$[c^4p^2\nabla^2 - (\varepsilon - V)^2\partial^2/\partial t^2]\,\Psi = 0\,, \tag{6}$$

where $\boldsymbol{p}$ is no longer constant. This generalization is justified by considering a small region in space over which V remains essentially constant. For such a local region the solution to eq. (6) becomes just a plane wave and the trajectories straight lines as in eq. (5).

In terms of the particle velocity, eq. (6) becomes

$$(v^2\nabla^2 - \partial^2/\partial t^2)\,\Psi = 0\,. \tag{7}$$

2·2. *Particle trajectories*. – The velocity of a point of constant phase and, therefore, the velocity of the particle traveling along a traveling-wave normal fixed to a surface of constant phase is given by

$$\boldsymbol{v} = -v^2\nabla\Psi/(\partial\Psi/\partial t)\,. \tag{8}$$

This specification of the trajectories, eq. (8), is unambiguous as long as traveling waves Ψ_t are being considered where

$$\Psi_t = \Psi[\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{v}t)]\,, \tag{9}$$

where $\boldsymbol{k}$ is the propagation constant and $\boldsymbol{v}(\boldsymbol{k}\cdot\boldsymbol{v}) = v^2\boldsymbol{k}$. However, when there are standing waves and the wave function is expressed as a product of a space function and a time function,

$$\Psi_s = \psi(\boldsymbol{r})\,T(t)\,, \tag{10}$$

it is not immediately apparent as to which direction along the wave normal the particle is traveling.

To resolve the difficulty two long wave trains traveling in opposite directions with the same wavelengths and speed may be considered. Substituting the resulting wave, $\Psi_s = \Psi_t(t) + \Psi_t(-t)$, into eq. (8), including an appropriate ambiguity of sign,

$$\pm\,\boldsymbol{v}(\boldsymbol{k}\cdot\boldsymbol{v})[-\Psi_t'(t) + \Psi_t'(-t)] + v^2\boldsymbol{k}[\Psi_t'(t) + \Psi_t'(-t)] = 0\,, \tag{11}$$

where the primes denote differentiation with respect to the argument as shown in eq. (9). In order for $\boldsymbol{v}$ to be positive and associated with the positive traveling wave it is necessary to choose the minus sign in eq. (11), since for $\boldsymbol{v}(\boldsymbol{k}\cdot\boldsymbol{v}) = v^2\boldsymbol{k}$ eq. (11) then yields $\Psi_t'[\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{v}t)] = 0$, or $\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{v}t) = \text{constant}$ on a surface of constant phase. Changing the signs of $\boldsymbol{k}$ and $\boldsymbol{v}$ also yields consistency for a negative phase velocity when the minus sign is chosen in eq. (11). A consistent definition of the phase velocity in a standing wave, thus, becomes

$$\boldsymbol{v} = v^2\nabla\Psi/(\partial\Psi/\partial t)\,, \tag{12}$$

which is identical to eq. (8) except for a change of sign.

2·3. *Hamilton-Jacobi equation*. – HAMILTON showed that light may be interpreted as a flux of particles traveling along the ray path acted on by a potential determined by the index of refraction. Here it is proposed to investigate the inverse problem of showing how a flux of classical nonrelativistic particles might exhibit wave properties.

In conformity with the velocity of energy and momentum transfer, the wave to be associated with the particles will be assumed to have a wave velocity equal to the particle velocity. In conformity with HAMILTON, the ray path is assumed to correspond to the particle trajectories. In particular, a wave is to be found with a phase velocity equal to the particle velocity which yields the Hamilton-Jacobi equation in the geometrical-optics limit. Considering the case where energy is conserved, the desired wave is

$$\Psi(\boldsymbol{r}, t) = \sin K[S_1(\boldsymbol{r}) - A(t)]\,, \tag{13}$$

where K is a large arbitrary constant, $S_1(\boldsymbol{r})$ is the Hamilton characteristic function which is expressed in terms of the position only and the constants of the motion, and $A(t)$ is the action expressed in terms of the time only and the constants of the motion.

Substituting eq. (13) into eq. (7) and preserving terms in K^2 only, the geometrical-optics approximation yields

$$v^2(\nabla S_1)^2 - (\partial A/\partial t)^2 = 0\,. \tag{14}$$

This result reduces to the Hamilton-Jacobi differential equation (8), since by definition

$$A(t) = \int_{t_0}^{t} \boldsymbol{p}\cdot\boldsymbol{v}\,\mathrm{d}t\,, \tag{15}$$

and $(\partial A/\partial t)^2 = (\boldsymbol{p}\cdot\boldsymbol{v})^2 = 2mv^2(E-V)$.

From the fact that [8] $\boldsymbol{p} = \nabla S_1$ and $\boldsymbol{p}\cdot\boldsymbol{v} = \partial A/\partial t$, it may be seen that eq. (8) specifies the correct particle velocity.

To briefly illustrate the principles involved, a free particle has $S_1(\boldsymbol{r}) = \boldsymbol{p}\cdot\boldsymbol{r}$ and $A(t) = \boldsymbol{p}\cdot\boldsymbol{v}t$ where $\boldsymbol{p}$ and $\boldsymbol{v}$ are constants of the motion. The traveling wave specified by eq. (13) then becomes

$$\Psi(\boldsymbol{r}, t) = \sin K[\boldsymbol{p}\cdot(\boldsymbol{r} - \boldsymbol{v}t)]\,, \tag{16}$$

(cf. eq. (1)).

This analysis shows that a flux of particles can follow classical trajectories while at the same time exhibiting possible wave behaviour, providing only that the wavelength, $2\pi/K|\boldsymbol{p}|$, is small compared with the smallest observed path length.

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(8) A. G. WEBSTER: *The Dynamics of Particles and of Rigid, Elastic and Fluid Bodies* (New York, 1949), p. 136.

The standing wave

$$\Psi(\boldsymbol{r}, t) = \sin[KS_1(\boldsymbol{r})]\sin[KA(t)]\,, \tag{17}$$

also yields the Hamilton-Jacobi eq. (14) in the geometrical-optics approximation. For this standing wave, eq. (17), the velocity of the particle is prescribed by eq. (12); thus,

$$\boldsymbol{v} = v^2 \frac{\nabla S_1(\boldsymbol{r})}{\partial A(t)/\partial t} \frac{\operatorname{ctg}[KS_1(\boldsymbol{r})]}{\operatorname{ctg}[KA(t)]}\,. \tag{18}$$

Multiplying eq. (18) by $\mathrm{d}\boldsymbol{r}/\mathrm{d}t$ and integrating yields

$$\sin[KA(t)]/\sin[KA(t_0)] = \sin[KS_1(\boldsymbol{r})]/\sin[KS_1(\boldsymbol{r}_0)]\,, \tag{19}$$

where the constants of integration have been chosen so that $S_1(r_0) = A(t_0)$ in conformity with classical theory. Neglecting the possible additive constant $2\pi n/K$, where n is an integer, which is presumably very small in any case, eq. (19) yields the classical prescription of the motion $S_1(r) = A(t)$. An equally valid classical solution is obtained by replacing the initial conditions with a minus sign, $S_1(r_0) = -A(t_0)$, which then specifies motion in the opposite direction along the trajectory. The generation of the two possible solutions was to be expected for the standing-wave case.

3. – Waves and quantum-mechanical trajectories.

The purely formal mathematical waves of Sect. 2 which were used to generate classical motion are now generalized to include quantum-mechanical motion.

3·1. *Postulates for quantum-mechanical motion.* – It is postulated, in the manner of SCHRÖDINGER [9], that any solution of the wave equations (6) or (7) subject to the usual boundary conditions that define a wave may yield a physically meaningful generating function Ψ. The quantum conditions, which arise as a consequence of the imposition of boundary conditions, are conditions that restrict the choice of some of the classical constants of the motion. The quantum conditions are evidence of the classical constants of the motion. The quantum conditions are evidence of the physical effect of fields or boundaries upon the particle's motion.

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[9] E. SCHRÖDINGER: *Ann. d. Phys.*, **79**, 361, 489 (1926); **80**, 437 (1926); **81**, 109 (1926).

In order for a particle to follow a nonclassical trajectory and thereby to exhibit the actual physical properties of a wave, the particle velocity as given by eqs. (8) and (12) may be generalized as follows: The velocity $\boldsymbol{v}$ appearing on the left of eqs. (8) and (12) is now interpreted as the instantaneous particle velocity, $\mathrm{d}\boldsymbol{r}/\mathrm{d}t$; while v^2 appearing on the right is interpreted as just the classical expression appearing in the wave equation (7). The integration is then carried out without necessarily restricting the constants of integration to the classical values. The motion so prescribed will, then, in general, be periodic with the arbitrary constant K, appearing in eqs. (13) an (17), entering into the results explicitly. Comparing eqs. (16) and (1), it is seen that experimental observation requires $K = 1/\hbar$. (cf. (9)).

3·2. *Old quantum theory.* – The successes of the old quantum theory of BOHR (10) and SOMMERFELD (11) are not properly taken into account by the traditional theory; since the old quantum theory permitted a completely causal view with point particles following discrete classical trajectories, while the traditional theory rejects such a view. The theory proposed here reduces to the old quantum theory for the geometrical-optics approximation. In order for the geometrical-optics approximation, eq. (17), to represent a standing wave, it is necessary for the period of the classical motion to be compatible with the period of the wave, or $KA(t_1) = 2\pi n$, where n is an integer and where $A(t_1)$ is the action evaluated over a period of the motion, eq. (15), where $t_1 - t_0$ is the classical period of the motion. But $KA(t_1) = 2\pi n$ is just the old quantum condition, where $K = 1/\hbar$.

3·3. *Separation of the wave equation into equations in space and time.* – For stationary waves $\Psi(\boldsymbol{r}, t)$ may be written as the product $\psi(\boldsymbol{r})T(t)$, eq. (10). Postulating that the space function $\psi(\boldsymbol{r})$ satisfies the time-independent Klein-Gordon equation

$$[c^2\hbar^2\nabla^2 + (\varepsilon - V)^2 - m^2c^4]\psi = 0\,, \tag{20}$$

or the time-independent Schrödinger equation, which may be derived from eq. (20) in the nonrelativistic limit,

$$[\hbar^2\nabla^2 + 2m(E - V)]\psi = 0\,, \tag{21}$$

then from eqs. (10) and (6) the time function $T(t)$ is found to satisfy

$$[\hbar^2\partial^2/\partial t^2 + (\boldsymbol{p}\cdot\boldsymbol{v})^2]T = 0\,, \tag{22}$$

where $(\boldsymbol{p}\cdot\boldsymbol{v})^2$ is the classical value expressed in terms of the time.

(10) N. BOHR: *Phil. Mag.*, **26**, 1 (1913).
(11) A. SOMMERFELD: *Ann. d. Phys.*, **51**, 1 (1916).

The quantum conditions remain the same as in the traditional theory, since these conditions depend only upon eqs. (20) or (21) an the boundary conditions. The time function $T(t)$, representing periodic motion, is not required to satisfy initial and final conditions; so that it yields no quantum conditions.

3·4. *Boundaries only.* – For a particle moving under the influence of boundaries only no potential need be explicitly included in eqs. (20) or (21); and $\boldsymbol{p}\cdot\boldsymbol{v}$, which is to be given the classical value in eq. (22) is a constant of the motion. The time function $T(t)$, satisfying eq. (22), becomes simply

$$T(t) = \sin(\boldsymbol{p}\cdot\boldsymbol{v}t/\hbar) = \sin[\varepsilon(1-m^2c^4/\varepsilon^2)t/\hbar] \tag{23}$$

This result, eq. (23), yields the traditional result for photons, but yields twice the frequency of the traditional nonrelativistic theory.

3·5. *Observed particle distributions.* – Defining the quantities $\boldsymbol{S}$ and P,

$$\begin{cases} \boldsymbol{S} = (\hbar/m)(\Psi_2\nabla\Psi_1 - \Psi_1\nabla\Psi_2)\,, \\ P = (\hbar/mv^2)(\Psi_1\partial\Psi_2/\partial t - \Psi_2\partial\Psi_1/\partial t)\,, \end{cases} \tag{24}$$

where Ψ_1 and Ψ_2 are two linearly independent solutions of the wave equation (7) and assuming that v^2 is not an explicit function of the time, it may be seen from eq. (7) that they satisfy the equation of continuity

$$\partial P/\partial t + \nabla\cdot\boldsymbol{S} = 0\,. \tag{25}$$

For a free-space traveling wave two linearly independent solutions of eq. (7) are

$$\Psi_{\substack{1\\2}} = a\,{}^{\sin}_{\cos}[\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{v}t)/\hbar]\,. \tag{26}$$

Substituting eqs. (26) into eqs. (24), the quantities $\boldsymbol{S}$ and P become

$$\boldsymbol{S} = (\boldsymbol{p}/m)a^2\,, \qquad P = (\boldsymbol{p}\cdot\boldsymbol{v}/mv^2)a^2\,. \tag{27}$$

Since a^2 is Lorentz-invariant and since $\boldsymbol{S} = \boldsymbol{v}P$, it is possible to associate P with the observed particle density and $\boldsymbol{S}$ with the observed particle flux (cf. traditional theory [12]). The fact that Lorentz invariance of the phase in eq. (26) is limited to values of $\boldsymbol{r}$ and t on the particle trajectory does not restrict the complete covariance of the resulting particle flux-density defined by $\boldsymbol{S}$ and P eq. (27).

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[12] L. I. SCHIFF: *Quantum Mechanics* (New York, 1949), pp. 21, 307.

For the bound particle case the linearly independent solutions of eq. (7) are given by eq. (10) where only the time parts need be linearly independent; thus, $\Psi_1 = \psi(\boldsymbol{r})T_1(t)$ and $\Psi_2 = \psi(\boldsymbol{r})T_2(t)$. From eq. (22) the Wronskian of T_1 and T_2 is a constant, $(\boldsymbol{p}\cdot\boldsymbol{v})/\hbar$. The net (or time average) particle flux and particle density from eq. (24) then becomes

$$S = 0\,, \qquad P = (\boldsymbol{p}\cdot\boldsymbol{v}/mv^2)\psi^2\,, \tag{28}$$

in agreement with traditional theory. Observed particle distributions are, thus, compatible with the present causal interpretation; and an assumption of inherent indeterminacy is not required, in agreement with BOHM ([13]) and DE BROGLIE ([14]).

3·6. *Motion in one dimension.* – Assuming that the solutions to eqs. (22) and (20) or (21) are known, the velocity of a bound particle may be obtained from eq. (12). Multiplying both sides of eq. (12) by the known classical expression $(\boldsymbol{p}\cdot\boldsymbol{v})^2 = (v/c)^2[(\varepsilon - V)^2 - m^2c^4]$, replacing $\boldsymbol{r}$ by x, and rearranging, the specification of th trajectory in one dimension becomes

$$[(\varepsilon - V)^2 - m^2c^4][\psi/(\mathrm{d}\psi/\mathrm{d}x)]\,\mathrm{d}x = [(\boldsymbol{p}\cdot\boldsymbol{v})^2 T/(\mathrm{d}T/\mathrm{d}t)]\,\mathrm{d}t\,. \tag{29}$$

From eqs. (22) and (20) or (21) it may be seen that eq. (29) is immediately integrable, yielding the desired trajectory

$$\psi'(x)/\psi'(x_0) = T'(t)/T'(t_0)\,, \tag{30}$$

where primes denote differentiation.

If only an approximate solution to eq. (22) is known, then eq. (30) cannot be used and eq. (29) must be integrated directly, since the aproximate solution is not, in fact, a solution to the differential equation. Thus, for the interesting geometrical-optics approximation, $T \approx \sin[A(t)/\hbar]$, were $A(t)$ is the classical action, eq. (15), the approximate trajectory from eq. (29) becomes

$$\psi'(x) \approx \psi'(x_0)\cos[A(t)/\hbar]\,, \tag{31}$$

where $A(t_0)$ has been chosen equal to zero or some integral multiple of $2\pi\hbar$.

([13]) D. BOHM: *Causality and Chance in Modern Physics* (Princeton, N. J. 1957).

([14]) L. DE BROGLIE: *Nonlinear Wave Mechanics, a Causal Interpretation* (Amsterdam, 1960).

4. – Examples.

4·1. *Free particle.* – In contrast to the traditional quantum theory a free particle need not be represented here by a unique wave function such as eq. (1). For example the wave function for a particle in a box [this example was presented in a previous paper (3)] yields the trajectory of a free particle when the boundaries of the box are removed to infinity and the initial conditions are appropriately chosen. The simplest traveling-wave function for a free particle, satisfying eq. (7), is

$$\Psi = x - v_0 t\,, \tag{32}$$

where v_0 is a constant. It may be seen by substituting eq. (32) into eq. (8) and integrating that eq. (32) does, in fact, yield the trajectory of a free particle. Moreover, any function of the argument given by eq. (9) also represents a free particle.

If no boundary conditions are imposed a standing wave may also represent a free particle. The simplest such solution of the form specified by eq. (10) and satisfying eq. (7) is given by

$$\Psi = (x - x_0)\,t\,, \tag{33}$$

where x_0 is a constant. Substituting eq. (33) into eq. (12) and integrating yields

$$x = x_0 \pm vt\,, \tag{34}$$

where $x = x_0$ when $t = 0$ and v is a constant. The two possible directions of motion indicated in eq. (34) were to be expected for a particle trajectory represented by a standing wave.

4·2. *Reflection from a plane mirror.* – To obtain the complete trajectory of a particle reflected from a plane mirror, not only in the incidence region and in the region of interference, but also in the region of reflection, would require the consideration of waves of finite lateral extent as stressed by DE BROGLIE (14). Here only the motion in the region of interference will be considered. Co-ordinates and geometry are chosen as indicated in Fig. 1.

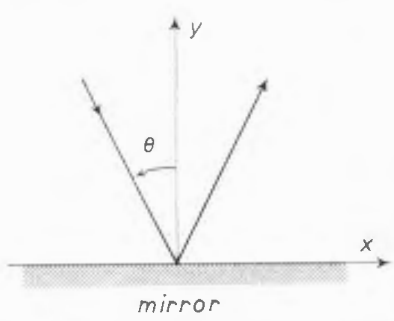

Fig. 1. – Co-ordinates and geometry for a particle reflected from a plane mirror.

Adding the incident and reflected plane waves of the form specified by eq. (1), the wave function for a nonrelativistic particle in the region of interference becomes

$$\Psi = 2\sin(p_y y/\hbar)\sin[(p_x x - 2Et)/\hbar]\,; \tag{35}$$

where the phases have been chosen so that Ψ vanishes on the mirror and where $p_x = p\sin\theta$, and $p_y = p\cos\theta$.

The velocity of the particle is specified by eq. (8) for the unbound motion in the x direction and by eq. (12) for the bound motion in the y direction. Substituting eq. (35)

into (8) and integrating,

$$x = x_0 + p_x t/m \, . \tag{36}$$

Substituting eq. (35) into eq. (12) gives

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{p_y}{m}\,\frac{\operatorname{ctg}(p_y y/\hbar)}{\operatorname{ctg}[(p_x x - 2Et)/\hbar]} \, . \tag{37}$$

Substituting x as given by eq. (36) into eq. (37) and integrating, the motion in the y direction is given by

$$\cos(p_y y/\hbar) = \cos(p_y y_0/\hbar)\cos[(p_x x_0 - p_y^2 t/m)/\hbar] \, . \tag{38}$$

This result, eq. (38), prescribes the same motion as that of a particle in a box (3) except that p_y^2/m is not quantized. Superimposed upon the uniform motion in the x direction, eq. (36), is the oscillatory motion in the y direction, eq. (38), as diagramed in Fig. 2. These results may be compared with the horizontal straight line trajectory found by DE BROGLIE (14).

From eqs. (26), (27) and (35) the naturally occurring particle density is found to correspond to the intensity of Weiner fringes.

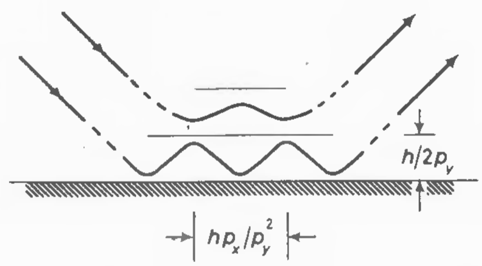

Fig. 2. – Two possible trajectories for a particle reflected from a mirror. The dashed line portions represent the uncertain transition from free waves to interfering waves.

4·3. *Simple harmonic oscillator.* – The function $\psi(x)$ for the nonrelativistic case is given by Schrödinger's equation (21), and the time function $T(t)$ is given by the differential eq. (22). Substituting the classical potential energy, $V = m\omega^2 x^2/2$, where ω is a constant, into Schrödinger's equation (21) yields the eigenvalues $E = (n + \frac{1}{2})\hbar\omega$ where n is an integer when the ψ function is assumed to vanish at infinity. The space function becomes

$$\psi_n = \exp[-y^2/2]\,He_n(y) \, , \tag{39}$$

where y is the numerical distance, $y = (m\omega/\hbar)^{\frac{1}{2}} x$, and He_n is the Hermite polynomial (15).

From the classical expression

$$\boldsymbol{p}\cdot\boldsymbol{v} = 2E\sin^2\omega t \, , \tag{40}$$

the differential equation (22) for the time function becomes

$$(\hbar^2\,\partial^2/\partial t^2 + 4E^2\sin^4\omega t)\,T = 0 \, . \tag{41}$$

This differential equation (41) is a special case of Hill's general differential equation.

(15) L. PAULING and E. B. WILSON: *Introduction to Quantum Mechanics* (New York, 1935), p. 67.

Here it will be sufficient to consider the geometrical-optics approximation which is valid for $2E/\hbar\omega \gg 1$ or $2n+1 \gg 1$. In this case the approximate solution of eq. (41) from $T \approx \sin[A(t)/\hbar]$, (15), and (40) becomes

$$T \approx \sin[(E/2\hbar\omega)(2\omega t - \sin 2\omega t)]\,, \tag{42}$$

the choice of the sine as opposed to the cosine being arbitrary. In the neighborhoods of the maxima and minima eq. (42) becomes exact. Ignoring the slow oscillations associated with the classical motion the rapid quantum-mechanical motion represented by eq. (42) becomes, on the average,

$$T \approx \sin(Et/\hbar)\,, \tag{43}$$

which is the time variation assumed in the traditional theory. This average frequency in eq. (43) is one half the frequency associated with a free particle, eq. (1) in the nonrelativistic limit.

Substituting eq. (42) into eq. (31) the trajectory of the particle is approximately

$$\psi_n'(y) = \psi_n'(y_0)\cos[(E/2\hbar\omega)(2\omega t - \sin 2\omega t)]\,, \tag{44}$$

where primes denote differentiation and where $\psi_n(y)$ is given by eq. (39). The motion prescribed by eq. (44) may be conveniently analysed by resorting to the classical device of a fictitious potential defined by

$$U \equiv E - m\dot{x}^2/2\,; \tag{45}$$

thus,

$$U \approx E\left\{1 - [\psi_n'^2(y_0) - \psi_n'^2(y)]/(2n+1)\,\psi_n^2(y)\right\}. \tag{46}$$

This function is shown on the left in Fig. 3 for $n=0$, 1 and 2 and various initial positions y_0. Like the problem of a particle in a box [3] the motion is cellular, the particle being confined to one of $n+1$ possible cells.

The inside cell boundaries, $y=a_j$, where the fictitious potential, eq. (46) is positively infinite, are given by the zeros of ψ_n or from eq. (39) by the zeros of the Hermite polynomial, $He_n(a_j)=0$ where $j\leqslant n$. Differing from the particle in a box, the outside cells are not bounded on the outside by an infinite fictitious potential. Nevertheless, there is still a maximum choice for y_0. For values of y_0 greater than this maximum the fictitious potential remains less than unity for all values of $y \geqslant y_0$, so that the particle leaves for infinity. From eq. (46) it may be seen that this critical value of y_0 is given for ψ_n' a maximum or for $\psi_n''=0$. From Schrödinger's equation (21) this means for $(E-V)\psi_n=0$; but since $\psi_n\neq 0$ for this region, the outside boundary is given by $E-V=0$, or just the classical displacement. This result, which is compatible with macroscopic observations, differs from the traditional quantum theory which allows the particle to range to infinity.

Like the particle in a box each cell contains a point of stagnation where the particle remains at rest if it is initially there with zero velocity. This compares with the stagnation DE BROGLIE [14] obtains for *all* bound particle trajectories. From eq. (44) these points of stagnation, $y=y_0=b_j$, are given by $\psi_n'(b_j)=0$, where $j\leqslant n$. As was to be expected these points occur at the maxima of ψ_n^2.

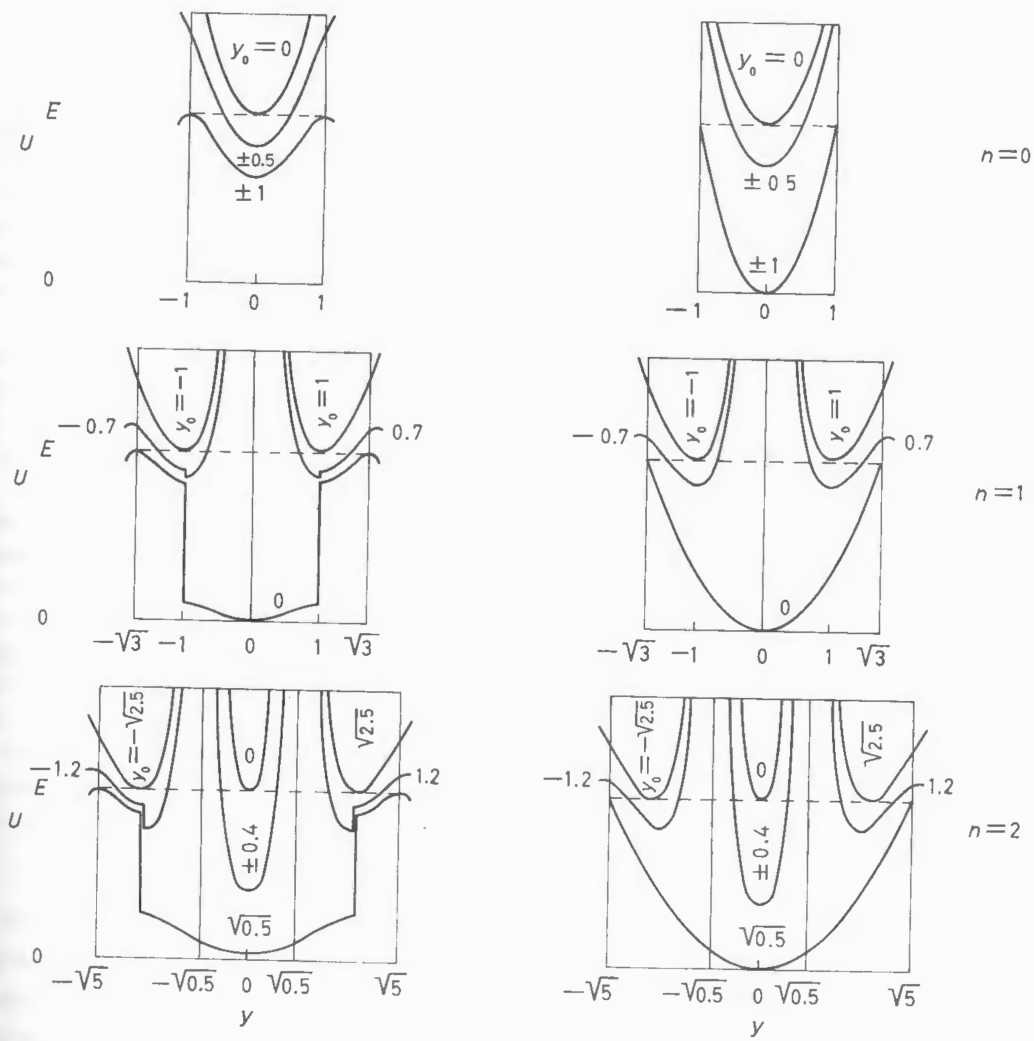

Fig. 3. – Approximate fictitious potential U, eq. (45), for the simple harmonic oscillator for $n = 0,1$ and 2 and various initial positions. On the left is plotted eq. (46) for various initial positions. On the right are curves sketched to eliminate the artificial jump discontinuities and to show the classical potential for coalescence. The uppermost curves show stagnation.

The turning points of the motion as deduced from eqs. (44) and (46), which occur for $U = E$, are y_0 and y_0' where $|y_0'| \geqslant |y_0|$ and y_0' is in the same cell as y_0. If eq. (46) were exact, y_0' could be obtained from y_0 by setting $\psi'(y_0') = \psi'(y_0)$; but due to the approximate nature of eq. (46) it is necessary to introduce an approximate matching of turning points,

$$(47) \qquad (b_j^2 - y_0^2)/(b_j^2 - a_j^2) = (y_j'^2 - b_j^2)/(a_{j+1}^2 - b_j^2)\,,$$

the points being ordered as follows:

$$(48) \qquad 0 < a_j < y_0 < b_j < y_0' < a_{j+1}\,,$$

the reverse order being taken for points to the left of the origin. This choice of y_0', eq. (47), introduces a jump discontinuity in the fictitious potential (indicated on the left in Fig. 3) but it permits all points to be accessible to the particle. The curves on the right of Fig. 3 have been sketched to eliminate this artificial jump discontinuity.

Since macroscopically a particle is observed to cross cell boundaries as it travels back and forth, there must be some choice of the initial conditions that permit this motion. As in the case of a particle in a box (3), if the initial position y_0 is taken on a cell boundary a_j, then the particle will be able to pass from cell to cell and macroscopic motion will result. In the limit as $y \to a_j$ and $y_0 = a_j$, using l'Hospital's rule and Schrödinger's equation (21), the approximate fictitious potential eq. (46) reduces to the classical potential, $U \approx V(a_j)$. The inside cell boundaries, thus, no longer have an infinite fictitious potential for this choice of y_0, and all of the cells coalesce. The lowest curves on the right in Fig. 3 show the classical potential energy as an approximation for the coalescence of the cells.

The uppermost curves for stagnation are exact and are the same on the right as on the left in Fig. 3.

Macroscopically the period is just the sum of the periods for each of the $n+1$ cells (considering coalescence). From eq. (43) for the period of the average cell the macroscopic period becomes

$$(n+1)(2\pi\hbar/E) = 2\pi(n+1)/(n+\tfrac{1}{2})\omega\,, \tag{49}$$

which is seen to be the correct classical period as $n \to \infty$.

5. – Discussion.

When boundaries are removed to infinity, when potentials are allowed to go to zero, or when the mass of a particle becomes macroscopically large, the theory proposed here yields a correspondence between bound quantum-mechanical motion and free classical particle motion (for the appropriate initial conditions). Not only does the traditional theory fail to establish such a necessary correspondence, but the causal theories of de Broglie and Bohm also fail (bound particles never move) as pointed out by EINSTEIN (16-17).

The present theory displays the known empirical facts in a self-cosistent classical causal framework. There has been no attempt to speculate about possible underlying physical mechanisms that could cause boundaries and potential fields to produce the observed quantum mechanical motion.

The prediction of new experimental results may be possible. For example, a particle executing simple harmonic motion is confined within the classical limits according to the theory presented here in contrast to the traditional theory which permits the particle to range to infinity. Consequently, there

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(16) A. EINSTEIN: in *Scientific Papers Presented to Max Born* (New York, 1953), p. 33.

(17) K. R. POPPER: *Logic of Scientific Discovery* (New York, 1961), p. 448.

should be a small difference in the normalization constant predicted by the two theories. This small difference in the normalized wave functions means that there may be a small difference predicted for transition probabilities and thus, a small difference predicted for the intensities of lines in vibrational spectra.

RIASSUNTO (*)

Si propone una teoria relativistica della meccanica quantica causale per una particella scalare, in cui la velocità di fase è uguale alla velocità della particella. Si considera la particella come una particella puntiforme classica che si muove lungo una normale all'onda tangente ad una superficie di fase costante. Si deduce una condizione per le traiettorie della particella in onda stazionaria che nel caso della particella legata dà origine a moti periodici. Nell'approssimazione dell'ottica geometrica si ottiene la vecchia teoria quantica di Bohr e Sommerfeld. La densità osservata della particella è l'intensità dell'onda (in accordo col $\Psi\Psi^*$ della teoria tradizionale). Si studiano i problemi di una particella libera, di una particella riflessa da uno specchio e dell'oscillatore armonico semplice.

(*) *Traduzione a cura della Redazione.*