Diffusion of Seismic Energy in the Near Range

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Abstract. It is assumed that the flow of seismic energy from an underground nuclear explosion or earthquake in the near range (i.e., distances of less than 1000 km) may be estimated by the diffusion equation in cylindrical geometry with a diffusivity varying directly as frequency. A term is included to allow for the dissipation of energy due to the anelasticity of the earth. The mean period and amplitude of a seismogram are thereby derived as functions of the time of arrival and range. The peak particle velocity is derived as a function of range. A preliminary comparison with observations of the seismic waves produced by an underground nuclear explosion indicates satisfactory over-all agreement.

INTRODUCTION

The present theory concerning the diffusion of seismic energy is proposed in order to obtain an over-all description of a seismogram in the near range. An adequate over-all description of a seismogram should prove useful for estimating the probable peak particle motion to be expected in the neighborhood of an underground nuclear explosion. Such a description might also yield some insight into the general mechanisms of seismic wave propagation.

For the near range (i.e., distances of less than 1000 km) the seismograms are quite different from those for greater ranges [cf., *Carpenter*, 1965; *Romney*, 1959]. In the near range, *P*, *S*, and surface waves of short periods are superimposed to yield extremely complicated seismograms. Farther out from the source the seismograms become simpler; the dominant period becomes longer and the various propagation modes become separated in time.

An extensive literature exists today covering various aspects of seismic waves [cf., *Miklowitz*, 1960]; much theoretical work has been done on the propagation of elastic waves in layered mediums [e.g., *Cagniard*, 1962; *Brekhovskikh*, 1960; *Ewing et al.*, 1957]; and a considerable amount of detailed information has been amassed under the heading of *seismology* [e.g., references in *Bullen*, 1963]; but there appear to be no formulas in the literature which properly describe the gross over-all characteristics of a seismogram in the near range. In particular, there appear to be no simple formulas that properly give the variation of the mean period and amplitude as functions of time of arrival and range (discrete arrivals and changes of phase being ignored).

THEORY

In the near range the dominant periods are sufficiently short for elastic waves to respond to numerous inhomogeneities in the earth. The propagation of elastic waves thus takes on the character of diffusion. At larger ranges (greater than about 1000 km) where the dominant periods become large and the elastic waves no longer respond to many inhomogeneities, the propagation of elastic waves in the earth may no longer be characterized by a diffusion process.

Diffusion equation. As is well known, the shape of an elastic pulse is propagated without change in an ideal infinite, homogeneous, nondispersive, elastic medium. In the heterogeneous earth, however, a single pulse produced by an underground nuclear explosion is rapidly converted into a long train of pulses and transients with a resulting apparent periodicity. The gross features of the seismogram may be attributed to numerous reflections and refractions. Each reflection and refraction at an interface gives rise to new compressional and shear waves (and possibly surface modes) which travel at different velocities. The number of possible new ray paths created at each interface is in general four: refracted compressional, refracted shear, reflected compressional, and reflected shear. The number of possible ray paths in the heterogeneous earth rapidly becomes quite large, and the over-all description of the phenomenon approaches that of a diffusion process. It is thus postulated that the seismic energy density E is specified approximately by the diffusion equation [*Joos*, 1934],

$$\partial E/\partial t - k\nabla^2 E = 0 \qquad (1)$$

where t is the time and k is the diffusivity.

Diffusivity as a function of frequency. The precise way in which k varies with the frequency is an extremely difficult theoretical question (cf. diffusion processes discussed in *Morse and Feshbach* [1953]). To fulfill the primary object of the present work, which is to develop a formula that fits the over-all observations, it is found that k should be chosen to vary directly with frequency:

$$k = R_m^2\omega/4 \qquad (2)$$

where ω is the angular frequency and R_m is a parameter with the dimensions of length (which may be compared with a scattering length or mean free path). A proper theoretical justification for this choice based upon first principles cannot be given at this time.

Cylindrical geometry. Since the velocity of seismic waves near the earth's surface generally increases with depth, seismic waves initially proceeding in a downward direction will tend to be refracted back toward the surface. The energy of the seismic waves appears to be largely confined to the outermost layers of the earth. For the near range being considered here the curvature of the earth may be ignored and the diffusion equation (1) may, thus, be appropriately solved in cylindrical coordinates.

Energy dissipation due to the anelasticity of the earth. Except at high frequencies where Rayleigh scattering may take place in a granular medium [e.g., *Knopoff and Porter*, 1963; *Mason and McSkimm*, 1947] or in metals where heat conduction produces a nonadiabatic loss [*Zener*, 1948], real solids generally do not behave in a viscoelastic manner. Neither a Voigt solid nor a Maxwell solid or, in fact, any combination of the two fits the empirical facts [*Kolsky*, 1953]. In a real solid at low frequencies a hysteresis loop is traced out, the area of which is independent of the frequency [*Love*, 1927; *Gemant and Jackson*, 1937; *Wegel and Walther*, 1935; *Taylor*, 1946; *Kolsky*, 1958; *Davids*, 1960]. This means that the fractional energy lost per unit time is proportional to the frequency. For a particular wave mode (such as S or P) the energy decreases with the residence time of the wave in the medium according to

$$\exp(-\omega t/Q) \qquad (3)$$

where the parameter Q is the usual dimensionless constant defined to characterize the medium [cf. *Anderson and Archambeau*, 1964].

For the heterogeneous earth the energy loss for a wave following a particular ray path becomes $\exp(-\omega\Sigma_i t_i/Q_i)$, where i refers to different portions of the path and the total time of residence is $t = \Sigma_i t_i$. An average value $\langle Q\rangle^{-1}$ may be defined by

$$\langle Q\rangle^{-1} = (\sum t_i Q_i^{-1})/\sum t_i \qquad (4)$$

Expression 3 may be regarded as valid for the heterogeneous earth if Q^{-1} is interpreted as an average value in the sense of (4). This average may be assumed to be taken over all modes of propagation as well as all ray paths.

The diffusion equation (1) may now be modified to include this energy dissipation due to the anelasticity of the earth. From (1), (2), and (3) the appropriate differential equation becomes

$$\partial E/\partial(\omega t) = (R_m^2/4)\nabla^2 E - E/Q \qquad (5)$$

It may be noted that, although the loss due to anelasticity is empirically correct, it has never been justified by a proper theoretical analysis based upon first principles.

Solution of diffusion equation with source. Because cylindrical geometry is assumed for a plane earth and diffusion takes place parallel to the plane boundary, the solution of (5) may be chosen as that for an unbounded medium. It may be assumed that the energy from an underground nuclear explosion or earthquake is injected into the earth as a δ function in time. The source may be approximated by a line source for points of observation sufficiently far removed. Introducing a δ function line source into (5) yields the appropriate diffusion equation with source included:

$$\partial E/\partial(\omega t) = (R_m^2/4)\nabla^2 E - E/Q - \pi R_m^2 B\delta(t)\delta(R) \qquad (6)$$

where R is the radial distance from the site of the explosion or earthquake, B is a constant specifying the strength of the source, and the time of the explosion or earthquake is $t = 0$. The appropriate solution to (6) [*Morse and Feshbach*, 1953] is

$$E = Bt^{-1} \exp(-R^2/R_m^2\omega t - \omega t/Q) \quad (7)$$

where B may be a function of the frequency.

This result specifies the energy density per unit frequency interval. To obtain the total energy density E_{tot} by summing over all frequencies it is only necessary to integrate (7); thus

$$E_{tot} = \frac{1}{t}\int_0^{\omega_0} B(\omega) \cdot \exp\left(-\frac{R^2}{R_m^2\omega t} - \frac{\omega t}{Q}\right) d\omega \quad (8)$$

a finite range of frequencies being assumed. The energy, not the particle velocity (or seismic wave amplitude), is summed over all frequencies because the individual waves of various frequencies are assumed to arrive at the point of observation with random phases. As is well known, the net intensity or energy flux of waves added together from sources with random phases is just the sum of the intensities produced by each of the sources individually.

Source function $B(\omega)$. To specify the function $B(\omega)$ we must know the nature of the original source. If the original source is approximated by a particle displacement that becomes established stepwise in time (as is frequently assumed, e.g., *Cagniard* [1962]), the Fourier transform of the particle displacement varies as ω^{-1} and the transform of the particle velocity is a constant independent of ω. If the original source is approximated by a particle displacement that becomes established as a δ function in time (as assumed to derive the Green's function for the displacement field [*Morse*, 1958]), the Fourier transform of the particle displacement is a constant and the transform of the velocity varies as ω.

Since the energy density per unit frequency interval E varies as the square of the velocity transform, B in (7) is a constant for a step function source and varies as ω^2 for the δ function source. It was found that the δ function source not only yields a better fit with the observed data but appears to yield a more physically satisfactory solution in general. In particular, a step function source fails to give meaningful results when the shear modulus is allowed to go to zero, even for pure P waves. It is thus assumed that B may be approximated by

$$B = B_0\omega^2 \quad (9)$$

Period as a function of time and range. The apparent or dominant frequency of a seismogram should correspond to the frequency of the maximum seismic energy density. Substituting (9) into (7), differentiating with respect to ω, and setting the result equal to zero, we find the apparent period to be a linear function of the time of arrival t:

$$T = St \quad (10)$$

(see Figure 2) where the slope S is a function of the range R,

$$S = (2\pi R_m^2/R^2)[(R^2/R_m^2Q + 1)^{1/2} - 1] \quad (11)$$

This slope decreases with increasing range.

Representation of a sinusoidal wave with a period proportional to the time. A simple periodic expression such as in $(2\pi t/T)$ ceases to be a periodic function—or, indeed, any function of the time at all—if T, as given by (10) is substituted directly into such an expression. The sinusoidal representation of a wave whose period is governed by (10) must be represented by some more general expression, $\sin \phi(t)$. For this expression to be a proper single-valued periodic function of the time t, the phase $\phi(t)$ should be a monotonically increasing function of the time. If the period of the wave is only a slowly increasing function of the time, the slope of ϕ as a function of t should be proportional to $2\pi/T$. From (10) this means that ϕ should satisfy

$$d\phi/dt = (2\pi/S)t^{-1} \quad (12)$$

Integrating (12) yields

$$\phi = (2\pi/S) \ln(t/t_0) \quad (13)$$

where t_0 is a constant of integration. The sinusoidal representation of a wave whose period is proportional to the time, according to (10), then becomes

$$\sin[(2\pi/S) \ln(t/t_0)] \quad (14)$$

Particle velocity amplitude as a function of *time of arrival.* Because (14) represents the sinusoidal variation of the seismic wave of maximum energy density, the dominant frequency as a function of t and R is given by

$$\omega^* = (2\pi/St) \ln (t/t_0) \qquad (15)$$

If this value of ω is chosen as a mean value with which to evaluate the integral in (8), according to the mean value theorem for integrals, the total energy density summed over all frequencies may be approximated by

$$E_{\mathrm{tot}} = \omega_0 E(\omega^*) \qquad (16)$$

Because the total seismic energy density varies as the square of the particle velocity, the velocity amplitude A is proportional to the square root of E_{tot}, or

$$A = A_0 \omega^* t^{-1/2} \cdot \exp(-R^2/2R_m^2\omega^* t - \omega^* t/2Q) \qquad (17)$$

where A_0 is a constant.

To facilitate comparison with observation it is convenient to introduce a numerical range ρ and the logarithmic time τ such that

$$\rho = R/Q^{1/2}R_m \qquad \tau = \ln (t/t_0) \qquad (18)$$

From (11) the slope S may be written

$$S = (2\pi/Q)[(\rho^2 + 1)^{1/2} + 1]^{-1} \qquad (19)$$

and (17) becomes

$$A = C \exp \{\ln \tau - 5\tau/2 - \tfrac{1}{2}[(\rho^2 + 1)^{1/2} - 1](\tau + \tau^{-1})\} \qquad (20)$$

(see Figure 3) where C is a function of the range:

$$C = A_0 t_0^{-3/2} Q[(\rho^2 + 1)^{1/2} + 1] \qquad (21)$$

For the time $t = t_0$, $\tau = 0$, and according to (20) $A = 0$; thus the parameter t_0 is the initial arrival time, zero time being the time of the underground nuclear explosion or earthquake.

Theoretical seismogram. According to the present theory the seismogram representing mean particle velocity v may be synthesized by multiplying (20) and (14); thus

$$v = A \sin [(2\pi/S) \ln (t/t_0)] \qquad (22)$$

(see Figure 1), where the random variations in amplitude and phase which occur in actual seismograms are not represented. This synthesis of the seismogram (22) is valid only under the assumption that the amplitude and the phase remain essentially constant over a single cycle, in order that the square of (22) averaged over a cycle should be proportional to the energy density (i.e., the amplitude squared). The average amplitude and phase of actual seismograms (as indicated by the theoretical curve in Figure 1) remain essentially constant over a single cycle, so that the synthesis (22) is quite adequate.

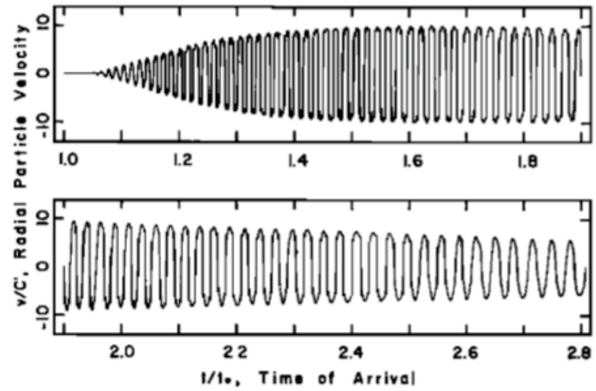

Fig. 1. Theoretical seismogram of the particle velocity (22) which duplicates the observed variation of average amplitude and period with time of arrival. Random variations of amplitude and phase such as occur in an actual seismogram are absent.

According to the present over-all view, it may be assumed that the heterogeneous earth partitions the energy equally among the three independent modes of particle motion along three mutually perpendicular axes. Consequently, (22) may also be used to represent the over-all vertical, transverse, or radial components of the particle velocity, the amplitude being smaller by the factor $3^{-1/2}$.

Peak particle velocity as a function of range. Differentiating the velocity amplitude (20) with respect to τ and setting the result equal to zero yields the value of τ for which A is a maximum; thus

$$\tau_m = [4 + (\rho^2 + 1)^{1/2}]^{-1} \cdot \{1 + [\rho^2 - 2 + 3(\rho^2 + 1)^{1/2}]^{1/2}\} \qquad (23)$$

The initial arrival time t_0 may be assumed to be a linear function of the range:

$$t_0 = R/c \qquad (24)$$

where c is the velocity of this first arrival. Therefore, from (18) and (21), C becomes

$$C = C'\rho^{-3/2}[(\rho^2 + 1)^{1/2} + 1] \quad (25)$$

where the new constant C' is not a function of the range,

$$C' = A_0 c^{3/2} Q^{1/4} R_m^{-3/2} \quad (26)$$

Substituting τ_m for τ and (25) for C into (20) yields the peak particle velocity:

$$v_p = C' \exp \{-(3/2) \ln \rho + \ln [(\rho^2 + 1)^{1/2} + 1] + \ln \tau_m - 5\tau_m/2 - \tfrac{1}{2}[(\rho^2 + 1)^{1/2} - 1](\tau_m + \tau_m^{-1})\} \quad (27)$$

(see Figure 4).

It may be noted that τ_m is not a widely varying function of ρ, since for $0 \leq \rho < \infty$, $2/5 \leq \tau_m \leq 1$; so that the variation of the peak particle velocity with numerical range is not particularly sensitive to the values of τ_m. For the extreme cases of $\rho \to 0$ and $\rho \to \infty$ it is found from (27) that

$$\begin{aligned} v_p &\approx \rho^{-3/2} \quad &\text{for} \quad \rho \to 0 \\ v_p &\approx \rho^{-1/2} e^{-\rho} \quad &\text{for} \quad \rho \to \infty \end{aligned} \quad (28)$$

where the numerical range ρ is defined by (18).

Comparison with Observation

To compare theory with observation the seismic waves produced by the underground nuclear explosion Klickitat, exploded at 15 hours 30 minutes and 0.1 seconds on February 20, 1964, at the Nevada test site (37°9′3.00″N, 116°2′24.00″ W) in tuff (yield classified) were observed at eight separate stations. Measurements were taken by the Coast and Geodetic Survey along an approximate line west from the epicenter at six stations designated B–1 through B–6 at the distances indicated in Table 1. United Electrodynamics, Inc., provided seismograms recorded at two stations, KN–UT (Kanab, Utah) and DR–CO (Durango, Colorado), at the distances indicated in Table 1. At stations B–1 and B–2 the signals were observed with standard strong-motion instruments, Coast Survey accelerometers (accurate for frequencies less than 5 cps) and Carder displacement meters (accurate for frequencies greater than 0.5 cps). At stations B–3 through B–6 National Geophysical Company's NC–21 velocity meters were used which are linear from 1 to about 50 cps. At KN–UT and DR–CO portable Benioff horizontal seismometers (displacement meters), Geotech model 6102A at DR–CO and model 1101 at KN–UT, whose responses are not flat at any frequency, were used. Readings from B–1 through B–6 were frequency corrected below 1 cps. Readings from KN–UT and DR–CO were frequency corrected at all frequencies. The records at B–1 through B–6 were recorded on moving photographic paper. The records from KN–UT and DR–CO were recorded on magnetic tape and later plotted on paper for visual inspection.

TABLE 1. Observed Slope of Period as a Function of Arrival Time

Station	Range R, km	Slope S, $\times 10^{-3}$
B–1	14.52	26.3
B–2	24.59	16.1
B–3	43.31	13.9
B–4	69.59	9.70
B–5	92.79	13.40
B–6	105.5	9.68
KN–UT	286.0	5.23
DR–CO	731.7	4.93

According to the present theory the three vector components of the particle velocity should be equivalent because of the equipartition of energy in the heterogeneous earth; consequently, only the radial component (particle motion toward and away from the source) was chosen for study.

Observed period as a function of the time of arrival. An average frequency for a particular time was estimated at stations B–1 and B–2 by counting the number of maximums, minimums, and zeros per unit time on the seismogram for the radial displacement, adding the number of intersections per unit time with the zero axis on the seismogram of the radial acceleration, and dividing the total by 6. At stations B–3 through B–6 the average frequency was estimated by counting the maximums, minimums, and zeros per unit time on the seismogram of the radial velocity, adding the number of intersections with the zero axis on the seismogram of the radial displacement (obtained from the velocity record by integration on an electronic analog computer), and dividing the total by 6. At stations KN–UT and DR–CO the frequency was estimated by counting the number of maximums and minimums per unit time on the seismogram of the radial displacement and dividing by 2. The time intervals over which the counts were taken (and thus the frequencies averaged) were taken sufficiently large to include at least three complete cycles. This necessitated an increase in the time interval chosen from 1 second at the beginning of the records to 5 seconds after about 50 seconds.

The observed results are presented in Figure 2 as a scatter diagram of the period as a function of the time of arrival. The straight lines are least-squares fits. The observed slopes are presented in Table 1.

The values of the two constants (appearing in (11)) R_m and Q which characterize the earth were computed from the eight observed slopes as shown in Table 1 by taking a least-squares fit (weighting the value of each slope according to the number of data points that were available to determine the slope) of the straight line S^2R^2 plotted as a function of S. According to (11) the slope of this straight line should be $-4\pi R_m^2$ and the S^2R^2 intercept should be π/Q. In this way it was found that

$$R_m = 6.4 \pm 1.3 \text{ km} \qquad Q = 185 \pm 75 \tag{29}$$

The probable errors indicated in (29) are perhaps larger than need be indicated, since these same values of R_m and Q give good agreement with other types of independent observations as indicated in the following sections.

The failure of some of the curves in Figure 2 to pass through the origin remains unexplained.

Observed amplitude as a function of time of arrival. The amplitude of the radial velocity was estimated as one-half the maximum minus one-half the minimum in each of the time intervals for which the average frequency was estimated (an interval involving at least three complete cycles). For stations B–1 and B–2 the amplitude of the particle velocity A was estimated from the amplitude of the acceleration A_a and the amplitude of the displacement A_d by letting $A = \omega_d A_d + A_a/\omega_a$, where ω_a is the observed angular frequency in the time interval on the acceleration seismogram (taken as 2π times the number of intersections with the zero axis per unit time) and ω_d is the corresponding angular frequency for the displacement. The seismograms from stations B–1 and B–2 were only

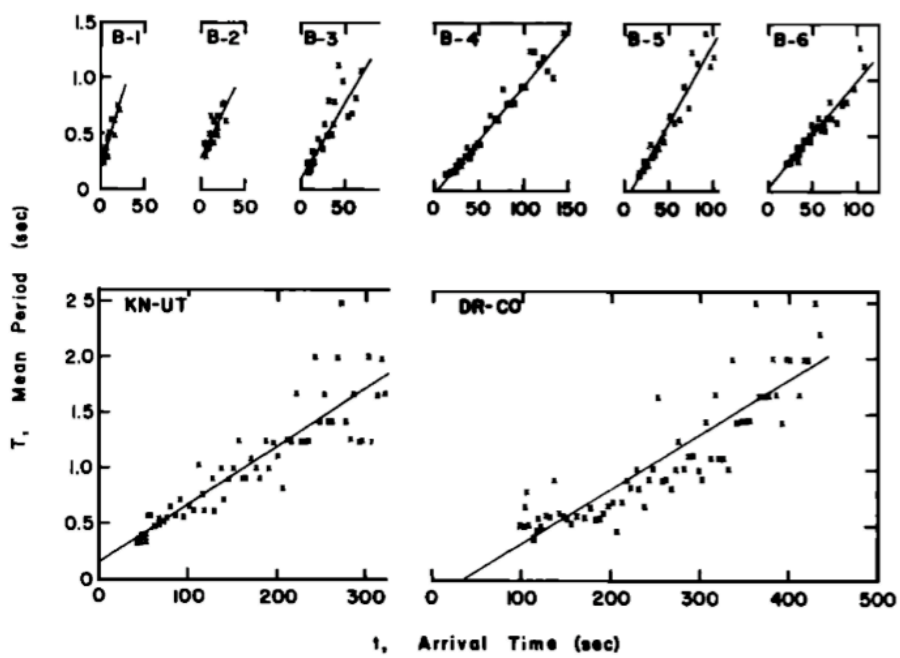

Fig. 2. Variation of period of the radial particle velocity with time of arrival. The straight lines, as required by (10), have been fitted by least squares.

30 seconds long and were, consequently, of limited value. At stations KN–UT and DR–CO the velocity amplitude was estimated from the displacement amplitude by letting $A = \omega_d A_d$.

The observed values of the amplitude A as functions of the time of arrival are shown on logarithmic scales in Figure 3. The theoretical curves shown in Figure 3 were obtained from (20) with the numerical range ρ computed from $Q^{1/2}R_m = 87$ km (according to (29) an error of $\pm$ 35 km may be associated with this value). The parameters t_0 and C were chosen for each individual curve in order to obtain the best fit. A change in t_0 means a back-and-forth translation of the curve on the logarithmic scale, while a change in C means an up-and-down translation, the shape of the curves remaining fixed.

Observed peak particle velocity as a function of range. The observed peak radial particle velocity v_p is shown in Figure 4. The theoretical curve was obtained from (27) by using the same value of $Q^{1/2}R_m = 87$ km to compute ρ as before. The value of C' was chosen to obtain the best fit.

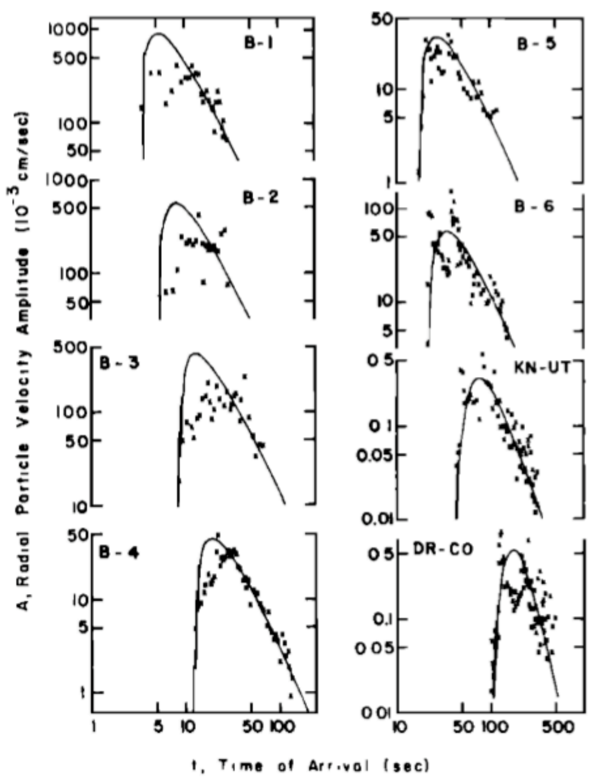

Fig. 3. Variation of amplitude of radial particle velocity with time of arrival, showing the theoretical curves (20).

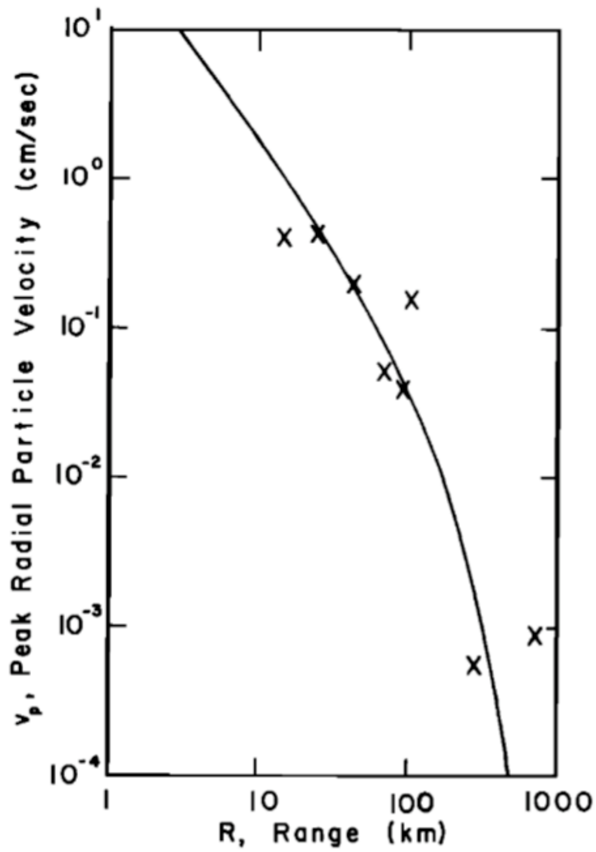

Fig. 4. Variation of peak radial particle velocity with range, showing the theoretical curve (27).

Conclusions

The theoretical assumption that the flow of seismic energy may be predicted by the diffusion equation with a diffusivity proportional to the frequency yields satisfactory over-all agreement with observation as indicated by Figures 1 through 4.

The linear relationship (10) between the period and the time of arrival as predicted by theory agrees with observation as shown in Figure 2. The decrease in the slope (11) of these straight lines with range is also in accord with observation (see Table 1).

The variation of the particle velocity amplitude (20) with the time of arrival agrees adequately with observations as shown in Figure 3. The best fit is for later times when the number of ray paths becomes large and the diffusion picture becomes more accurate. Since $R_m = 6.4$ km may be viewed as a scattering length, it would appear that the diffusion approximation may fail for ranges of this order of magnitude. Better agreement between theory and observation would probably be obtained if the rms value of the magnitude of the vector velocity were used for the amplitude instead of the peak velocity spread in each time interval for just the radial component.

The variation of the peak particle velocity with range as predicted by theory (27) appears to fit the data (see Figure 4) fairly well considering the scatter of the observed data points.

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