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OSCILLATING VERTICAL MAGNETIC DIPOLE ABOVE A CONDUCTING HALF-SPACE

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James Paul Wesley

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ABSTRACT

The electromagnetic field produced by a vertical oscillating magnetic dipole above a plane conducting earth is obtained in integral form. An exact solution in closed form is obtained for the dipole and the point of observation located on the surface of the earth.

* Work performed under the auspices of the U. S. Atomic Energy Commission.

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INTRODUCTION

The present work investigates the electromagnetic field produced by a vertical magnetic dipole above a plane conducting earth. For the case of especial interest, the dipole and the point of observation located on the interface between air and earth, the field components may be obtained in exact closed form for all ranges of the parameters.

The magnitudes of the parameters of interest here are specified as follows:

Frequency:	$\approx 10^4$ cycles/second.
Distance from dipole:	$<$ 4775 meters (wavelength in air).
Conductivity of earth:	10^{-2} mho/meter for damp earth to
	10^{-5} mho/meter for dry earth.

CHOICE OF COORDINATES

An oscillating vertical magnetic dipole of time variation $\exp(-i\omega t)$ is located, at a distance h, above a half-infinite conducting earth as shown in Fig. 1. Cylindrical coordinates are chosen with z directed upwards through the dipole, and the radius ρ taken horizontally and in the plane of the surface of the earth. The air is characterized by the parameters ϵ_0, μ_0, and $\sigma_1 = 0$, while the earth is characterized by ϵ_0, μ_0, and $\sigma_2 = \sigma \neq 0$ where rationalized mks units are assumed.

DERIVATION OF THE DIFFERENTIAL EQUATION

Introducing the wave number, k,

$$k^2 = \omega^2 \mu\epsilon + i\omega\mu\sigma, \tag{1}$$

and the intrinsic admittance, η,

$$k\eta = \omega\epsilon + i\sigma, \tag{2}$$

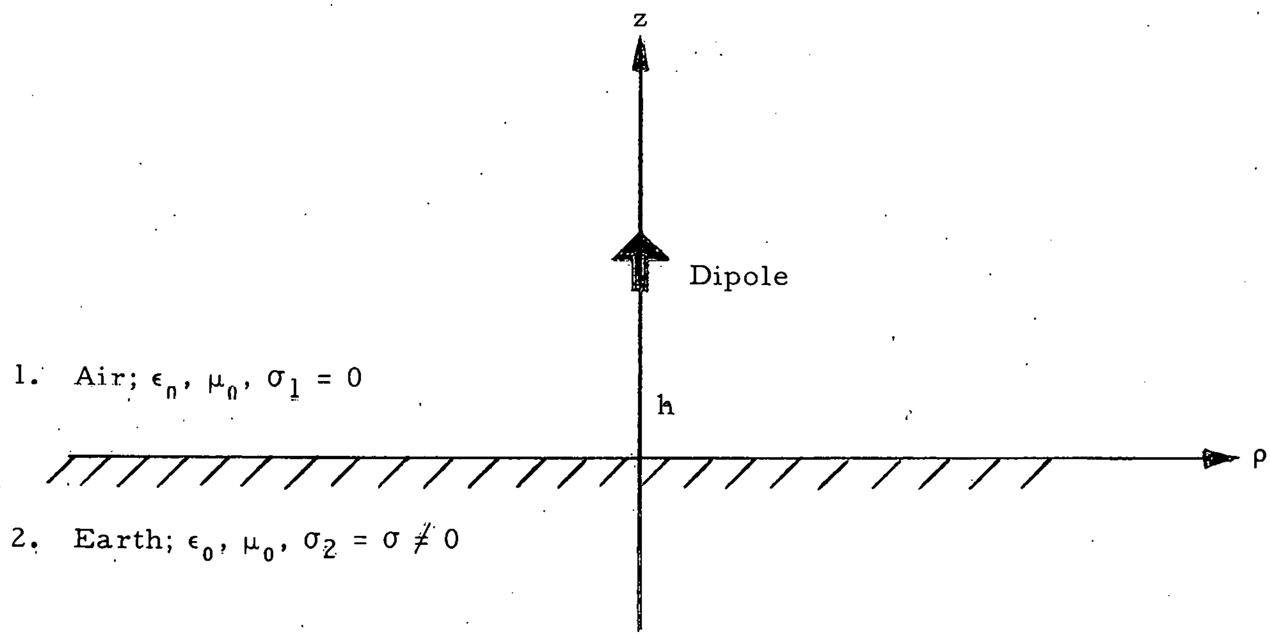

Fig. 1. Choice of coordinates for the magnetic dipole placed a distance h above a plane conducting earth.

one may write Maxwell's equations in rationalized mks units as follows:

$$\begin{aligned} &\nabla \times \underset{\sim}{E} = ik\,\underset{\sim}{H}, &\quad& \nabla \cdot \underset{\sim}{H} = 0, \\ &\nabla \times \underset{\sim}{H} = -ik\eta\,\underset{\sim}{E} + \underset{\sim}{J}^0, &\quad& \nabla \cdot \underset{\sim}{E} = \nabla \cdot \underset{\sim}{J}^0/ik\eta. \end{aligned} \tag{3}$$

When combined with the Hertzian vector potential

$$\underset{\sim}{E} = \nabla\nabla \cdot \underset{\sim}{\pi} + k^2 \underset{\sim}{\pi}, \qquad \underset{\sim}{H} = -ik\eta\,\nabla \times \underset{\sim}{\pi}, \tag{4}$$

Maxwell's equations (3) yield the differential equation

$$\nabla \times \nabla \times \underset{\sim}{\pi} - \nabla\nabla \cdot \underset{\sim}{\pi} - k^2 \underset{\sim}{\pi} = -\underset{\sim}{J}^0/ik\eta. \tag{5}$$

The source current for a magnetic dipole may be specified as the current in a circular loop where the radius goes to zero as the current goes to infinity in such a way that the current times the area of the loop remains finite and nonzero. Thus,

$$\underset{\sim}{J}^0 = \lim_{\pi a^2 I \to m} \underset{\sim}{e}_\phi\, I\, \delta(\rho - a)\, \delta(z - h), \tag{6}$$

where $\delta(\rho - a)$ and $\delta(z - h)$ are the Dirac delta functions, I is the current, m is the magnetic dipole moment, and $\underset{\sim}{e}_\phi$ is a unit vector in the ϕ direction.

Since there is symmetry about the z axis, we may attempt a solution using only the ϕ component of the Hertz vector. Noting that nothing will be a function of the azimuthal angle ϕ, we have Eq. (5) yielding

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2} + k^2\right]\pi_\phi = J^0/ik\eta\,. \tag{7}$$

This differential Eq. (7) may be simplified by letting

$$\pi_\phi = \partial\Psi/\partial\rho. \tag{8}$$

Substituting Eqs. (8) and (6) into Eq. (7) then yields the desired differential equation

$$(\nabla^2 + k^2)\Psi = - \lim_{\pi a^2 I \to m} \frac{I}{ik\eta}\,\mathscr{A}(a - \rho)\,\delta(z - h), \tag{9}$$

where $\mathscr{A}(z - \rho)$ is the unit step function, zero for $\rho \geq a$.

SOLUTION TO THE INHOMOGENEOUS EQUATION

To solve Eq. (9) for the inhomogeneous part we may apply Green's theorem to Ψ and Φ in an infinite space where Φ is defined by

$$(\nabla^2 + k^2)\Phi = - \delta(\underset{\sim}{r} - \underset{\sim}{r}'), \tag{10}$$

where

$$\Phi = e^{ik|\underset{\sim}{r} - \underset{\sim}{r}'|}/4\pi|\underset{\sim}{r} - \underset{\sim}{r}'|\,. \tag{11}$$

We thus obtain

$$\int_V \left[\Phi(\nabla^2 + k^2)\Psi - \Psi(\nabla^2 + k^2)\Phi\right] dV = \int_S \left(\Phi\frac{\partial\Psi}{\partial n} - \Psi\frac{\partial\Phi}{\partial n}\right) dS = 0\,, \tag{12}$$

where Φ and $\partial\Phi/\partial n$ vanish on the sphere at infinity.

Substituting Eqs. (9), (10), and (11) into Eq. (12) then yields

$$\Psi(\underset{\sim}{r}') = \lim_{\pi a^2 I \to m} \frac{I}{4\pi ik\eta}\int_V \mathscr{A}(a - \rho)\,\delta(z - h)\,\frac{e^{ik|\underset{\sim}{r} - \underset{\sim}{r}'|}}{|\underset{\sim}{r} - \underset{\sim}{r}'|}\,dV\,. \tag{13}$$

Performing the z integration, we have

$$\Psi(\underset{\sim}{r}') \lim_{\pi a^2 I \to m} \frac{I}{4\pi i k \eta} \int_0^{2\pi} \int_0^{a} \rho \frac{e^{ikR'}}{R'} d\rho d\phi , \tag{14}$$

where

$$R'^2 = \rho'^2 + \rho^2 - 2\rho\rho' \cos(\phi' - \phi) + (z' - h)^2 . \tag{15}$$

In the limit as $a \to 0$ (such that as $I \to \infty$, $\pi a^2 I \to m$) we have by the mean value theorem for integrals

$$\Psi(\underset{\sim}{r}') = \lim_{\pi a^2 I \to m} \frac{I}{4\pi i k \eta} (a - 0) \frac{a}{2} \int_0^{2\pi} d\phi \frac{e^{ikR''}}{R''} , \tag{16}$$

where

$$R''^2 = \rho'^2 + (a/2)^2 - a\rho' \cos(\phi' - \phi) + (z' - h)^2 . \tag{17}$$

In the limit as $a \to 0$ we can perform the ϕ integration to obtain

$$\Psi(\underset{\sim}{r}) = \frac{m}{4\pi i k \eta} \frac{e^{ikR_1}}{R_1} , \tag{18}$$

where

$$R_1^2 = \rho^2 + (z - h)^2 . \tag{19}$$

From Eqs. (18) and (8) the Hertz vector then becomes

$$\pi_\phi^0 = \frac{m}{4\pi i k \eta} \frac{\partial}{\partial \rho} \frac{e^{ikR_1}}{R_1} , \tag{20}$$

where the superscript zero indicates that this is the solution to the inhomogeneous part of Eq. (7).

SOLUTION OF THE HOMOGENEOUS EQUATION

Using the Sommerfeld[1] representation we get the solution to the homogeneous part of Eq. (7), Eq. (8), by differentiating Eq. (9):

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[1] A. Sommerfeld, Ann. Physik 28, 665-737 (1909).

$$\pi_\phi^1 = - \frac{m}{4\pi ik\eta}\int_0^\infty f(\lambda)\, e^{-\gamma_z} J_1(\lambda\rho)\lambda^2\, d\lambda, \tag{21}$$

where

$$\gamma = (\lambda^2 - k^2), \tag{22}$$

such that

$$\lim_{\lambda\to 0} \gamma \to -ik. \tag{23}$$

INTEGRAL FORM OF THE SOLUTION

Using Sommerfeld's[1] integral representation

$$\frac{e^{ik_1R_1}}{R_1} = \int_0^\infty \frac{1}{\gamma_1} e^{-\gamma_1|z-h|} J_0(\lambda\rho)\lambda\, d\lambda, \tag{24}$$

where

$$\gamma_1 = (\lambda^2 - k_1^2)^{1/2}, \tag{25}$$

and introducing an image function which is a solution to the homogeneous Eq. (9), we have

$$-\frac{e^{ik_1R_2}}{R_2} = -\int_0^\infty \frac{1}{\gamma_1} e^{-\gamma_1(z+h)} J_0(\lambda\rho)\lambda\, d\lambda, \quad \text{for } z \geq 0, \tag{26}$$

where

$$R_2^2 = \rho^2 + (z+h)^2; \tag{27}$$

the Hertz vector in air, according to Eqs. (20) and (21), may be written in the form

$$\pi_{\phi 1} = - \frac{m}{4\pi ik_1\eta_1}\int_0^\infty \left\{\frac{1}{\gamma_1} e^{-\gamma_1|z-h|} - \frac{1}{\gamma_1} e^{-\gamma_1(z+h)} + f_1(\lambda)\, e^{-\gamma_1(z+h)}\right\} J_1(\lambda\rho)\lambda^2\, d\lambda, \tag{28}$$

for $z \geq 0$. In the conducting earth Eq. (21) yields

$$\pi_{\phi 2} = - \frac{m}{4\pi ik_2\eta_2}\int_0^\infty f_2(\lambda)\, e^{-\gamma_1 h+\gamma_2 z} J_1(\lambda\rho)\lambda^2\, d\lambda, \tag{29}$$

for $z \leq 0$ and where

$$\gamma_2 = (\lambda^2 - k_2^2)^{1/2}. \tag{30}$$

APPLICATION OF THE BOUNDARY CONDITIONS

From Eq. (4) and the assumption that only a ϕ component of the Hertz vector is needed, the field components are given by

$$H_z = -ik\eta \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\pi_\phi),$$

$$H_\rho = -ik\eta \frac{\partial}{\partial z}\pi_\phi, \tag{31}$$

$$E_\phi = k^2\pi_\phi .$$

For the continuity of the magnetic field across the boundary between earth and air and the continuity of the horizontal component of the electric field across the boundary, Eq. (30) yields the boundary conditions

$$k_1^2\pi_{\phi 1} = k_2^2\pi_{\phi 2},$$

$$k_1^2\frac{\partial\pi_{\phi 1}}{\partial z} = k_2^2\frac{\partial\pi_{\phi 2}}{\partial z}. \tag{32}$$

Substituting Eqs. (28) and (29) into Eqs. (32) then yields the requirements on $f_1(\lambda)$ and $f_2(\lambda)$,

$$f_1(\lambda) = f_2(\lambda),$$

$$-2 + \gamma_1 f_1(\lambda) = -\gamma_2 f_2(\lambda). \tag{33}$$

Solving Eqs. (33) yields

$$f_1(\lambda) = f_2(\lambda) = \frac{2}{\gamma_1 + \gamma_2}. \tag{34}$$

Substituting Eqs. (34) into Eqs. (28) and (29) then yields the result

$$\pi_{\phi 1} = \frac{m}{4\pi i k_1\eta_1}\frac{\partial}{\partial\rho}\left(\frac{e^{ik_1R_1}}{R_1} - \frac{e^{ik_1R_2}}{R_2} + \mathcal{U}_1\right),$$

$$\pi_{\phi 2} = \frac{m}{4\pi i k_2\eta_2}\frac{\partial}{\partial\rho}\mathcal{U}_2, \tag{35}$$

where

$$\mathcal{U}_1 = \int_0^\infty \frac{2}{\gamma_1 + \gamma_2} e^{-\gamma_1(z+h)} J_0(\lambda\rho)\lambda\, d\lambda, \qquad z \geq 0,$$

$$\mathcal{U}_2 = \int_0^\infty \frac{2}{\gamma_1 + \gamma_2} e^{-\gamma_1 h + \gamma_2 z} J_0(\lambda\rho)\lambda\, d\lambda, \qquad z \leq 0. \tag{36}$$

EVALUATION OF THE INTEGRALS

The problem has now been reduced to evaluating the integrals $\mathcal{U}_1$ and $\mathcal{U}_2$ in Eqs. (36), the Hertz vector and the field components being given by differentiation according to Eqs. (35) and (31). Both of the integrals $\mathcal{U}_1$ and $\mathcal{U}_2$ may be obtained from a single more general integral. First, we note that

$$\frac{2}{\gamma_1 + \gamma_2} \equiv - \frac{2}{k_1^2 - k_2^2}(\gamma_1 - \gamma_2), \tag{37}$$

using Eqs. (25) and (30). We thus define the general integral, J,

$$J \equiv 2\int_0^\infty e^{-\gamma_1 a - \gamma_2 b} J_0(\lambda\rho)\lambda\, d\lambda, \tag{38}$$

and obtain

$$\mathcal{U}_1 = \frac{1}{k_1^2 - k_2^2}\left[J_a - J_b\right]_{\substack{a=z+h\\ b=0}},$$

$$\mathcal{U}_2 = \frac{1}{k_1^2 - k_2^2}\left[J_a - J_b\right]_{\substack{a=h\\ b=-z}}, \tag{39}$$

where the subscripts on the J's denote differentiation. The problem has now been reduced to evaluating the general integral J, Eq. (38).

The evaluation of the integral J, Eq. (38), for various ranges of the pertinent parameters has been treated at some length in a previous work by Baños and Wesley.[2] Here we will limit our consideration to the case of

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[2] A. Baños and J. P. Wesley, "The Horizontal Electric Dipole in a Conducting Half-Space, Part II," Scripps Institute of Oceanography, La Jolla, California, Monograph SIO 54-31, 62 (1954).

both the source and the observer on the surface of the earth, $h = 0$, $z = 0$.

SPECIAL CASE $z = 0$, $h = 0$

The integrals may be evaluated in closed form for this special case where h and z are both zero, no approximations being necessary. The derivative of J, Eq. (38), with respect to a may be evaluated by first letting $b \to 0$; thus,

$$\lim_{a \to 0} J_a = \lim_{a \to 0} 2 \frac{\partial}{\partial a} \int_0^{\infty} e^{-\gamma_1 u} J_0(\lambda\rho)\lambda \, d\lambda. \tag{40}$$

Using the integral representation, Eq. (24), we then obtain for the right-hand side of Eq. (40),

$$-\lim_{a \to 0} 2 \frac{\partial^2}{\partial a^2} \frac{e^{ik_1 R}}{R}, \tag{41}$$

where

$$R^2 = \rho^2 + a^2. \tag{42}$$

Performing the indicated differentiations in (41) and going to the limit as $a \to 0$ then yields the result

$$\lim_{\substack{a \to 0 \\ b \to 0}} J_a = -\frac{2}{\rho} \frac{\partial}{\partial \rho} \frac{e^{ik_1 \rho}}{\rho}. \tag{43}$$

Using the same analysis for the derivative of J with respect to b, we get from Eqs. (39) the exact result for $h = 0$, $z = 0$,

$$\mathcal{U} = -\frac{2}{k_1^2 - k_2^2} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{e^{ik_1 \rho}}{\rho} - \frac{e^{ik_2 \rho}}{\rho} \right), \tag{44}$$

a result first obtained by van der Pol.[3] The subscript 1 or 2 on $\mathcal{U}$ has been

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[3] B. van der Pol, Z. Hochfrequenz-Technologie 37, 152-157 (1931).

dropped in Eq. (44), since on the surface there is no need to differentiate between medium 1 and medium 2, the value $\mathcal{U}$ being the same for both media.

FIELD COMPONENTS FOR h = 0, z = 0

To obtain the ρ component of the magnetic field, as given by the second of Eqs. (31), it is necessary to differentiate Eqs. (39) again with respect to a or b before letting a or b $\to 0$. Since even derivatives of J will vanish as a or b $\to 0$, the ρ component of the magnetic field is identically zero, a result which could have been anticipated from the symmetry.

From Eqs. (35) and (44) the Hertz vector becomes

$$\pi_{\phi 1} = -\frac{m}{4\pi i k_1 \eta_1} \frac{2}{k_1^2 - k_2^2} \frac{\partial}{\partial \rho}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{e^{ik_1\rho}}{\rho} - \frac{e^{ik_2\rho}}{\rho}\right)\right],$$

$$\pi_{\phi 2} = -\frac{m}{4\pi i k_2 \eta_2} \frac{2}{k_1^2 - k_2^2} \frac{\partial}{\partial \rho}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{e^{ik_1\rho}}{\rho} - \frac{e^{ik_2\rho}}{\rho}\right)\right]. \tag{45}$$

Performing the differentiations indicated in the first of Eqs. (45) yields

$$\pi_{\phi 1} = -\frac{m}{4\pi i k_1 \eta_1} \frac{2}{k_1^2 - k_2^2}\left\{\left(-\frac{k_1^2}{\rho^2} - \frac{3ik_1}{\rho^3} + \frac{3}{\rho^4}\right)e^{ik_1\rho} - \left(-\frac{k_2^2}{\rho^2} - \frac{3ik_1}{\rho^3} + \frac{3}{\rho^4}\right)e^{ik_2\rho}\right\}, \tag{46}$$

where a similar expression holds for $\pi_{\phi 2}$. It may be noted that as $k_1 \to k_2$, Eq. (46) reduces to the free-space expression, Eq. (20),

$$\pi_{\phi 1} = \frac{m}{4\pi i k \eta}\left(\frac{ik}{\rho} - \frac{1}{\rho^2}\right)e^{ik\rho}, \tag{47}$$

as it should.

Substituting Eq. (46) into the first and third of Eqs. (31) (the second being identically zero for h = 0, z = 0), and performing the indicated differentiations, we obtain the exact expressions for the field components for h = 0, z = 0,

$$H_z = \frac{m}{4\pi}\frac{2}{k_1^2 - k_2^2}\left\{\left(-\frac{ik_1^3}{\rho^2} + \frac{4k_1^2}{\rho^3} + \frac{9ik_1}{\rho^4} - \frac{9}{\rho^5}\right)e^{ik_1\rho} - \left(-\frac{ik_2^3}{\rho^2} + \frac{4k_2^2}{\rho^3} + \frac{9ik_2}{\rho^4} - \frac{9}{\rho^5}\right)e^{ik_2\rho}\right\};$$

$$E_\phi = \frac{m}{4\pi}\frac{2i\omega\mu}{k_1^2 - k_2^2}\left\{\left(-\frac{k_1^2}{\rho^2} - \frac{3ik_1}{\rho^3} + \frac{3}{\rho^4}\right)e^{ik_1\rho} - \left(-\frac{k_2^2}{\rho^2} - \frac{3ik_2}{\rho^3} + \frac{3}{\rho^4}\right)e^{ik_2\rho}\right\} \tag{48}$$

where it isn't necessary to distinguish between mediums 1 and 2 on the interface because of the continuity of H_z and E_ϕ.

FIELD COMPONENTS FOR $k_1\rho << 1$

For points of observation much less than a wavelength in air, the field components for the source and the point of observation on the surface of the earth, as given by Eqs. (48), become

$$H_z \approx \frac{m}{4\pi}\frac{2}{k_2^2}\left\{\frac{9}{\rho^5} + \left(-\frac{ik_2^3}{\rho^2} + \frac{4k_2^2}{\rho^3} + \frac{9ik_2}{\rho^4} - \frac{9}{\rho^5}\right)e^{ik_2\rho}\right\},$$

$$E_\phi \approx \frac{m}{4\pi}\frac{2i\omega\mu}{k_2^2}\left\{-\frac{3}{\rho^4} + \left(\frac{k_2^2}{\rho^2} - \frac{3ik_2}{\rho^3} + \frac{3}{\rho^4}\right)e^{ik_2\rho}\right\}. \tag{49}$$

FIELD COMPONENTS FOR $k_1\rho < |k_2\rho| << 1$

For points of observation less than a wavelength in air and also less than a wavelength in the conducting medium, we obtain the quasi- static limit,

$$H_z \approx -\frac{m}{4\pi}\frac{1}{\rho^3},$$

$$E_\phi \approx \frac{m}{4\pi}\frac{i\omega\mu}{\rho^2}, \tag{50}$$

for source and point of observation on the surface of the earth.

NUMERICAL WORK

For numerical work it is of interest to investigate the ranges of the parameters in some detail. From Eq. (1) the wave number, k, may be written as

$$k = \frac{1}{\delta}\left\{\left(\sqrt{1+\beta^2}+\beta\right)^{1/2} + i\left(\sqrt{1+\beta^2}-\beta\right)^{1/2}\right\}, \tag{51}$$

where δ is the skin depth,

$$\delta = (2/\omega\,\mu_0\sigma)^{1/2}, \tag{52}$$

and

$$\beta = \omega\,\epsilon_0/\sigma. \tag{53}$$

For the conducting earth we are interested in the case where $\sigma > 10^{-5}$ mho/meter and $\omega = 2\pi \times 10^4$ cycles per second. Since $\epsilon_0 = 8.854 \times 10^{-12}$ farad/meter, we may neglect $\beta^2 < 0.003$ as compared with unity. Thus, for the present case of interest, Eqs. (1) and (51) yield

$$\begin{aligned} k_1 &= \omega\sqrt{\mu_0\epsilon_0} = \omega/c = 1/\lambda_1, \\ k_2 &= (1+i)/\delta = (1+i)/\sqrt{2}\,\lambda_2. \end{aligned} \tag{54}$$

It is convenient to introduce the numerical distance x,

$$x \equiv \rho/\delta, \tag{55}$$

and the ratio

$$n \equiv \delta\omega/c = \sqrt{2}\;\lambda_2/\lambda_1, \tag{56}$$

where

$$0.01 < n < 0.3, \tag{57}$$

for $\omega = 2\pi \times 10^4$ cycles per second and $10^{-5} < \sigma < 10^{-2}$ mho/meter.

It is now possible to separate the exact result, Eqs. (48), into real and imaginary parts using Eqs. (54), (55), and (56). Thus we obtain

$$h_{zr} \equiv \mathrm{Re}\left\{\frac{2\pi}{m}\,\delta^3(n^4+4)H_z\right\}$$
$$= x^{-5}\left[A_1F(nx) + B_1\,G(nx) + e^{-x}C_1F(x) + e^{-x}D_1\,G(x)\right],$$

$$h_{zi} \equiv \mathrm{Im}\left\{\frac{2\pi}{m}\,\delta^3(n^4+4)H_z\right\}$$
$$= x^{-5}\left[A_1G(nx) - B_1\,F(nx) + e^{-x}C_1G(x) - e^{-x}D_1F(x)\right],$$

$$e_{\phi r} \equiv \mathrm{Re}\left\{\frac{2\pi}{m}\frac{\delta^2}{\omega\mu}(n^4+4)\,E_\phi\right\} \tag{58}$$
$$= x^{-4}\left[-A_2G(nx) + B_2\,F(nx) + e^{-x}C_2\,G(x) - e^{-x}D_2F(x)\right],$$

$$e_{\phi i} \equiv \mathrm{Im}\left\{\frac{2\pi}{m}\frac{\delta^2}{\omega\mu}(n^4+4)E_\phi\right\}$$
$$= x^{-4}\left[A_2F(nx) + B_2G(nx) - e^{-x}C_2F(x) - e^{-x}D_2G(x)\right],$$

where

$$F(x) = n^2\cos x - 2\sin x, \qquad G(x) = n^2\sin x + 2\cos x,$$
$$A_1 = 4n^2x^2 - 9, \qquad B_1 = n^3x^3 - 9nx,$$
$$C_1 = -2x^3 + 9x + 9, \qquad D_1 = 2x^3 + 8x^2 + 9x, \tag{59}$$
$$A_2 = -n^2x^2 + 3, \qquad B_2 = 3nx,$$
$$C_2 = 3x + 3, \qquad D_2 = 2x^2 + 3x.$$

For the case of points of observation much less than a wavelength in air, $k_1\rho \ll 1$, Eqs. (49) yield the simpler expressions:

$$h_{zr} \equiv \mathrm{Re}\left\{\frac{4\pi}{m}\delta^3 H_z\right\} \approx x^{-5}\, e^{-x}(A\cos x + B\sin x),$$

$$h_{zi} \equiv \mathrm{Im}\left\{\frac{4\pi}{m}\delta^3 H_z\right\} \approx -x^{-5}\left[9 + e^{-x}(B\cos x - A\sin x)\right],$$

$$e_{\phi r} \equiv \mathrm{Re}\left\{\frac{4\pi}{m}\frac{\delta^2}{\omega\mu}E\right\} \approx 3x^{-4}\left[-1 + e^{-x}(C\cos x + D\sin x)\right], \tag{60}$$

$$e_{\phi i} \equiv \mathrm{Im}\left\{\frac{4\pi}{m}\frac{\delta^2}{\omega\mu}E\right\} \approx 3x^{-4}\, e^{-x}(-D\cos x + C\sin x),$$

where

$$A = 2x^3 + 8x^2 + 9x, \qquad B = 2x^3 - 9x - 9,$$

$$C = x + 1, \qquad D = \frac{2}{3}x^2 + x. \tag{61}$$

CONCLUSIONS

A vertical oscillating magnetic dipole on the surface of an infinite plane conducting earth yields an electromagnetic field at the surface of the earth given exactly by Eqs. (48) for all ranges of the parameters. For points of observation less than a wavelength in the conducting earth away from the source, $k_1\rho < |k_2\rho| << 1$, Eqs. (50) show that the magnetic field varies as ρ^{-3} and the electric field varies as ρ^{-2}. For distances from the source greater than a wavelength in the conducting earth but less than a wavelength in air, $k_1\rho << 1 < |k_2\rho|$, Eqs. (49) show that the magnetic field varies as ρ^{-m} where $2 < m < 3$, and the electric field also varies as ρ^{-n} where $2 < n < 3$. For distances greater than a wavelength in air, both the electric and magnetic field components, Eqs. (48), vary as ρ^{-2}.

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