GEOPHYSICS, VOL. XXIII, NO. 1 (JANUARY, 1958), PP. 128–133, 1 FIG.

RESPONSE OF DYKE TO OSCILLATING DIPOLE*†

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ABSTRACT

A dyke of sulfide ore may be geophysically prospected by observing its electromagnetic response to a slowly oscillating magnetic dipole source. An excellent first approximation of the fields generated is obtained by considering the idealized case of a dyke of infinite conductivity and vanishing thickness in a vacuum. Surprisingly, this idealized problem can be solved exactly in terms of a newly discovered Green's function for Laplace's equation (in three dimensions) which is simply expressed in closed form. The magnetic scalar potential and the magnetic field are given for final results.

INTRODUCTION

Recent improvements in instrumentation make it possible to prospect geophysically a dyke of sulfide ore of high electrical conductivity by observing the response of the dyke to a slowly oscillating magnetic dipole source. For low frequencies (10 to 1,000 cps) the effect of the earth in which the dyke is actually imbedded may be neglected and the earth replaced by a vacuum. When the dyke appears to be geometrically thin, which will always be the case for source and observer far from the dyke, we may approximate the dyke of sulfide ore by a dyke of infinite conductivity and vanishing thickness.

Cylindrical coordinates are chosen as indicated in Figure 1, with the z-axis taken horizontally along the edge of the dyke, the dyke running from $z\to-\infty$ to $z\to+\infty$. The angle ϕ, which may be realized in the actual space, varies between $-\pi/2$ and $3\pi/2$. The radial distance from the edge of the dyke (the z-axis) is denoted by ρ. Cartesian coordinates are chosen with the x-axis horizontal and perpendicular to the dyke, and with the y-axis parallel to the dyke and vertical, the dyke extending from $y=0$ to $y\to-\infty$.

FUNDAMENTAL EQUATIONS

Maxwell's equations for a vacuum and for time harmonic solutions with the time variation $\exp(-i\omega t)$, where ω is the angular frequency, in rationalized *mks* units may be written according to Stratton (1941) in the form:

$$\begin{aligned} &\text{I.} \quad \nabla\times \boldsymbol{E} - i\omega\mu_0 \boldsymbol{H} = 0, \qquad &&\text{III.} \quad \nabla\cdot\boldsymbol{H} = 0 \\ &\text{II.} \quad \nabla\times\boldsymbol{H} + i\omega\epsilon_0\boldsymbol{E} = \boldsymbol{J}, \qquad &&\text{IV.} \quad \nabla\cdot\boldsymbol{E} = 0, \end{aligned} \tag{1}$$

where the symbols have their usual meaning. For the present very small frequencies we may take $\boldsymbol{E}=0$; and Maxwell's equations (1) reduce to

$$\nabla\times\boldsymbol{H} = \boldsymbol{J}, \qquad \nabla\cdot\boldsymbol{H} = 0. \tag{2}$$

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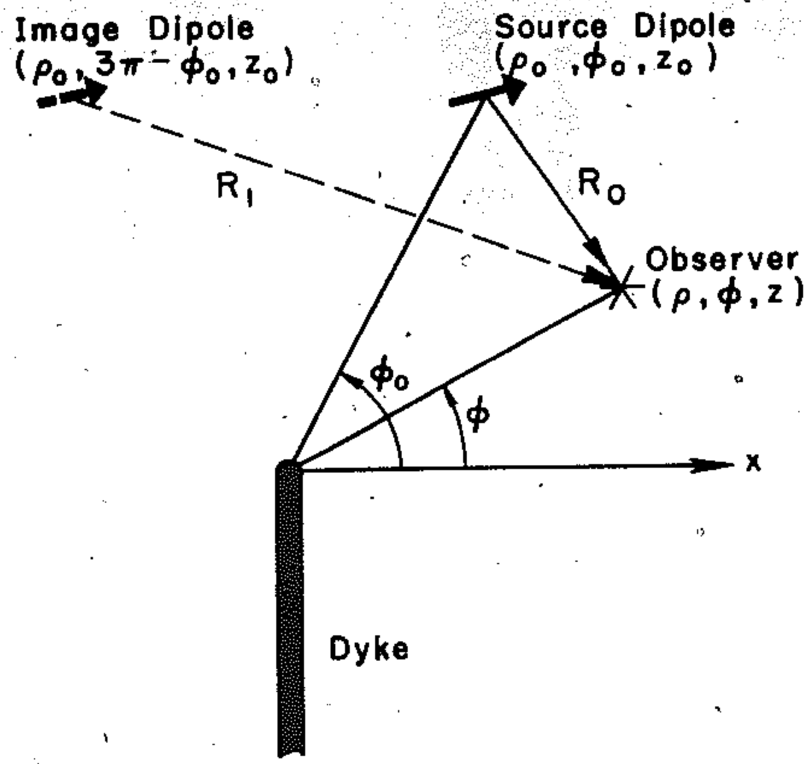

FIG. 1. Dyke of sulfide ore showing the choice of cylindrical coordinates with the z-axis taken along the edge of the dyke and perpendicular to the page. The vectors $\boldsymbol{R}_0=\boldsymbol{r}-\boldsymbol{r}_0$ from source to observer and $\boldsymbol{R}_1$ from image to observer are also indicated.

The prescribed current distribution J for the present magnetic dipole source is

$$J = \nabla \times \boldsymbol{m}\delta(\boldsymbol{r} - \boldsymbol{r}_0), \tag{3}$$

where $\delta(\boldsymbol{r}-\boldsymbol{r}_0)$ is the Dirac delta function, $\boldsymbol{r}$ is the position vector of the observer, $\boldsymbol{r}_0$ is the position vector of the source, and $\boldsymbol{m}$ is the magnetic dipole moment. Substituting equation (3) into the first of equations (2), we obtain,

$$H = \boldsymbol{m}\delta(\boldsymbol{r} - \boldsymbol{r}_0) - \nabla\psi, \tag{4}$$

where ψ is the scalar magnetic potential which from the second of equations (2) satisfies Poisson's equation,

$$\nabla^2\psi = \nabla\cdot\boldsymbol{m}\delta(\boldsymbol{r} - \boldsymbol{r}_0). \tag{5}$$

The boundary condition (Stratton, 1941, p. 37) requiring the normal component of the magnetic flux density to be continuous across a surface and the requirement that the magnetic field vanish inside the dyke of infinite conductivity, yields the pertinent boundary condition from equation (4),

$$\partial\psi/\partial n = 0 \quad \text{for} \quad \phi = -\pi/2,\ 3\pi/2, \tag{6}$$

where the derivative with respect to n means the normal derivative to the surface.

The problem is now to solve equations (5) and (6) for the geometry specified by Figure 1.

GREEN'S THEOREM

To solve equations (5) and (6) it is of interest to use Green's theorem (Morse and Feshback, 1953) for ψ and a Green's function specified by,

$$\nabla^2 G = -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}'),$$
$$\partial G/\partial n = 0 \quad \text{for} \quad \phi = -\pi/2,\ 3\pi/2. \tag{7}$$

Thus,

$$\int (G\nabla^2\psi - \psi\nabla^2 G)d\tau' = \int (G\partial\psi/\partial n - \psi\partial G/\partial n)da' = 0, \tag{8}$$

where the boundary conditions (6) and (7) have been used. Replacing the laplacian of ψ and G by the right hand member of equations (5) and (7), respectively, we obtain,

$$\int [G(\boldsymbol{r}, \boldsymbol{r}')\nabla'\cdot\boldsymbol{m}\delta(\boldsymbol{r}' - \boldsymbol{r}_0) + 4\pi\psi(\boldsymbol{r}')\delta(\boldsymbol{r} - \boldsymbol{r}')]d\tau' = 0. \tag{9}$$

The first integral in equation (9) may be evaluated by noting that

$$\int G(\boldsymbol{r}, \boldsymbol{r}')\nabla'\cdot\boldsymbol{m}\delta(\boldsymbol{r}' - \boldsymbol{r}_0)d\tau' = -\nabla^0\cdot\boldsymbol{m}\int G(\boldsymbol{r}, \boldsymbol{r}')\delta(\boldsymbol{r}' - \boldsymbol{r}_0)d\tau'$$
$$= -\nabla^0\cdot\boldsymbol{m}G(\boldsymbol{r}, \boldsymbol{r}_0), \tag{10}$$

where the superscript zero indicates differentiation with respect to the source coordinates. Integrating the second integral in equation (9) directly and using equation (10), we obtain the solution

$$\psi(\boldsymbol{r}) = \nabla^0\cdot\boldsymbol{m}G(\boldsymbol{r}, \boldsymbol{r}_0)/4\pi. \tag{11}$$

From equation (4), neglecting the case for which the source and observer are coincident, we have the desired magnetic field,

$$\boldsymbol{H} = -\nabla[\nabla^0\cdot\boldsymbol{m}G(\boldsymbol{r}, \boldsymbol{r}_0)/4\pi]. \tag{12}$$

GREEN'S FUNCTION

The Green's function[1] satisfying equations (7) may be written in closed form as the sum of a source term and an image term,

$$G = G_0 + G_1, \tag{13}$$

where

$$G_0 = (\pi + 2\tan^{-1} g_0/R_0)/2\pi R_0,$$
$$G_1 = (\pi + 2\tan^{-1} g_1/R_1)/2\pi R_1, \tag{14}$$

[1] The Green's function which vanishes on the surfaces $\phi = -\pi/2,\ 3\pi/2$, is given by $G = G_0 - G_1$.

where the values of the arctangent are always taken such that

$$-\pi/2 \leq \tan^{-1} g_0/R_0 \leq \pi/2, \qquad -\pi/2 \leq \tan^{-1} g_1/R_1 \leq \pi/2, \tag{15}$$

and where

$$\begin{aligned} g_0 &= 2\sqrt{\rho\rho_0}\cos \tfrac{1}{2}(\phi - \phi_0), \\ g_1 &= 2\sqrt{\rho\rho_0}\cos \tfrac{1}{2}(\phi + \phi_0 - 3\pi), \\ R_0^2 &= \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + (z - z_0)^2, \\ R_1^2 &= \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi + \phi_0 - 3\pi) + (z - z_0)^2. \end{aligned} \tag{16}$$

This new and particularly simple result, equations (13) through (16), may be derived by an involved direct analysis; it is better, however, to prove merely that this is the desired result.[2]

It may be verified by direct substitution that G_0 and G_1 separately and therefore G satisfy Laplace's equation. The boundary conditions appearing in equations (7) may be seen to be satisfied by noting that

$$\begin{aligned} G_0 &= G_0(\cos \tfrac{1}{2}(\phi + \phi_0)), \\ G_1 &= G_0(\cos \tfrac{1}{2}(\phi + \phi_0 - 3\pi)); \end{aligned} \tag{17}$$

and

$$\pm\, \partial G/\partial n = \partial G/\partial \phi = -\tfrac{1}{2}[\sin \tfrac{1}{2}(\phi - \phi_0) + \sin \tfrac{1}{2}(\phi + \phi_0 - 3\pi)]\, G_0' = 0 \quad \text{for } \phi = -\pi/2,\, 3\pi/2. \tag{18}$$

To prove that G satisfies the inhomogeneous part of the first of equations (7), we need only demonstrate that G varies as $1/R_0$ in the neighborhood of the source. Since $g_0 \sim 2\rho$ according to the first of equations (16) and $R_0 \sim 0$, we find from the first of equations (14) that $G_0 \sim 1/R_0$ as $R_0 \to 0$. And since R_1 remains finite, G does, in fact, vary as $1/R_0$ in the neighborhood of the source as it should. Finally, we note that the behavior at infinity is proper. As R_0 and therefore R_1 go to infinity we have $G_0 \sim 1/2R_0$ and $G_1 \sim 1/2R_1$ from equations (14). Since $R_1 \sim R_0$, we have $G \sim 1/2R_0 + 1/2R_1 \sim 1/R_0$, as it should.

It is of interest to note that the image function G_1, equation (14), remains finite even when the point of observation is coincident with the image point. This may be regarded as arising from the fact that the image is actually at the virtual point $\phi = 3\pi - \phi_0$ and is, therefore, not present in the real space where $-\pi/2 \leq \phi \leq 3\pi/2$.

[2] An interesting derivation of the present results has been obtained as a special case of the transient problem of a dipole established step-wise in time by J. R. Wait (1957) who had the present material by private communication.

MAGNETIC POTENTIAL

To compute the scalar magnetic potential we first write it as the sum of a source term and an image term as given by equations (11) and (13),

$$\psi = \psi_0 + \psi_1, \tag{19}$$

where

$$\psi_0 = \nabla^0 \cdot \boldsymbol{m} G_0/4\pi, \qquad \psi_1 = \nabla^0 \cdot \boldsymbol{m} G_1/4\pi. \tag{20}$$

Substituting the expressions for G_0 and G_1 as given by equation (14), (15), and (16) into equations (20), we obtain,

$$\begin{aligned} \psi_0 &= [\pi + 2 \tan^{-1} g_0/R_0 + 2g_0R_0/X^2]\boldsymbol{m} \cdot \nabla^0(1/8\pi^2 R_0) + (1/4\pi^2 X^2)\boldsymbol{m} \cdot \nabla^0 g_0, \\ \psi_1 &= [\pi + 2 \tan^{-1} g_1/R_1 + 2g_1R_1/X^2]\boldsymbol{m} \cdot \nabla^0(1/8\pi^2 R_1) + (1/4\pi^2 X^2)\boldsymbol{m} \cdot \nabla^0 g_1, \end{aligned} \tag{21}$$

where

$$X^2 = (\rho + \rho_0)^2 + (z - z_0)^2 = R_0^2 + g_0^2 = R_1^2 + g_1^2. \tag{22}$$

The derivatives appearing in equations (21) may be evaluated. In cartesian components we have

$$\begin{aligned} \nabla^0 g_0 &= (h_0 \mathbf{e}_x - g_1 \mathbf{e}_y)/2\rho_0, \\ \nabla^0 g_1 &= (h_1 \mathbf{e}_x - g_0 \mathbf{e}_y)/2\rho_0, \end{aligned} \tag{23}$$

where $\mathbf{e}_x$, $\mathbf{e}_y$ are unit vectors, g_0 and g_1 are given by the first two equations (16) and where

$$\begin{aligned} h_0 &= 2\sqrt{\rho\rho_0} \cos \tfrac{1}{2}(\phi + \phi_0), \\ h_1 &= 2\sqrt{\rho\rho_0} \cos \tfrac{1}{2}(\phi - \phi_0 \angle 3\pi); \end{aligned} \tag{24}$$

and

$$\begin{aligned} \nabla^0(1/R_0) &= \boldsymbol{R}_0/R_0^3, \\ \nabla^0(1/R_1) &= \boldsymbol{R}_1/R_1^3 - 2(x + x_0)\mathbf{e}_x/R_1^3, \end{aligned} \tag{25}$$

where the last two equations may be verified by noting that

$$\begin{aligned} R_0^2 &= (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2, \\ R_1^2 &= (x + x_0)^2 + (y - y_0)^2 + (z - z_0)^2. \end{aligned} \tag{26}$$

MAGNETIC FIELD

From equations (4) and (19), neglecting the case of coincident source and observer, we may write the magnetic field as the sum of a source term and an image term,

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_1 \tag{27}$$

where

$$\boldsymbol{H}_0 = -\nabla\psi_0, \qquad \boldsymbol{H}_1 = -\nabla\psi_1. \tag{28}$$

Substituting equations (21) through (25) into equations (28), we obtain

$$\begin{aligned} \boldsymbol{H}_0 = {} & [3\boldsymbol{m}\cdot\boldsymbol{R}_0\boldsymbol{R}_0 - \boldsymbol{m}R_0^2][\pi + 2\tan^{-1} g_0/R_0 + 2g_0R_0/X^2]/8\pi^2R_0^5 \\ & + \boldsymbol{R}_0\boldsymbol{m}\cdot[g_0\boldsymbol{R}_0 + R_0^2\nabla^0 g_0]/2\pi^2R_0^2X^4 \\ & - \boldsymbol{m}\cdot[\boldsymbol{R}_0 - g_0\nabla^0 g_0]\nabla g_0/2\pi^2X^4 - \nabla[\boldsymbol{m}\cdot\nabla^0 g_0]/4\pi^2X^2, \end{aligned} \tag{29}$$

$$\begin{aligned} \boldsymbol{H}_1 = {} & [3(\boldsymbol{m}\cdot\boldsymbol{R}_1 - 2(x + x_0)m_x)\boldsymbol{R}_1 - \boldsymbol{m}R_1^2 + 2m_xR_1^2\mathbf{e}_x] \\ & \times [\pi + 2\tan^{-1} g_1/R_1 + 2g_1R_1/X^2]/8\pi^2R_1^5 \\ & + \boldsymbol{R}_1\boldsymbol{m}\cdot[g_1(\boldsymbol{R}_1 - 2(x + x_0)\mathbf{e}_x) + R_1^2\nabla^0 g_1]/2\pi^2R_1^2X^4 \\ & - \boldsymbol{m}\cdot[\boldsymbol{R}_1 - 2(x + x_0)\mathbf{e}_x - g_1\nabla^0 g_1]\nabla g_1/2\pi^2X^4 \\ & - \nabla[\boldsymbol{m}\cdot\nabla^0 g_1]/4\pi^2X^2, \end{aligned} \tag{30}$$

where $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$ are defined by equation (16) or (26) and are shown in Figure 1, g_0 and g_1 are defined by equations (16), $\nabla^0 g_0$ and $\nabla^0 g_1$ are given by equations (23) and (24), the arctangents are confined to the range from $-\pi/2$ to $+\pi/2$, X^2 is given by equation (22),

$$\begin{aligned} \nabla g_0 &= (h_0\mathbf{e}_x - g_1\mathbf{e}_y)/2\rho, \\ \nabla g_1 &= -(h_1\mathbf{e}_x + g_0\mathbf{e}_y)/2\rho, \end{aligned} \tag{31}$$

where h_0 and h_1 are given by equations (24), and where

$$\begin{aligned} \nabla[\boldsymbol{m}\cdot\nabla^0 g_0] &= [-h_1\mathbf{e}_z \times \boldsymbol{m} + g_0(m_x\mathbf{e}_x + m_y\mathbf{e}_y)]/4\rho\rho_0, \\ \nabla[\boldsymbol{m}\cdot\nabla^0 g_1] &= [h_0\mathbf{e}_z \times (\boldsymbol{m} - 2m_x\mathbf{e}_x) + g_1(-m_x\mathbf{e}_x + m_y\mathbf{e}_y)]/4\rho\rho_0. \end{aligned} \tag{32}$$

CONCLUSIONS

The idealized problem of the electromagnetic response of a dyke of infinite conductivity and vanishing thickness to a slowly oscillating magnetic dipole can, surprisingly, be solved exactly in a simple closed form, equations (29) through (32). This result represents quite adequately the response of a dyke of large conductivity and small geometrical width. If the actual conductivity and width need to be known for purposes of identification of the ore and to estimate the size of the deposit, then it is necessary to refine the solution by approximate methods to be presented in a subsequent paper.

REFERENCES

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