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The Bethe–Weizsäcker Mass Formula and Lennard-Jones N–N Potentials

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An elementary derivation of the Bethe–Weizsäcker semiempirical nuclear mass formula which is in the spirit of current views of nuclear structure, is given. Lennard-Jones potentials are assumed to act between nucleons. Thus the major interaction between nn, pp, and np pairs is taken of the form $-g/r^3+h/r^4$, where r is the separation distance between nucleons, and g and h are constants. An additional "symmetry" interaction of the form $-s/r^3$ is assumed for np pairs. Summing the potential energy over all nucleon pairs and using the Fermi statistical estimate of the kinetic energy, the Bethe–Weizsäcker semiempirical mass formula is obtained directly. The constants of the mass formula are discussed in relation to the N–N interaction and are found to be quite plausible.

INTRODUCTION

The Bethe–Weizsäcker mass formula,[1,2]

$$E=-a_1A+a_2A^{2/3}+a_3Z^2/A^{1/3}+a_4D^2/A, \quad (1)$$

is well known as a simple and accurate representation of the systematics of nuclear energies.[3] Here A, Z, N, and $D(=N-Z)$ are the mass number, proton number, neutron number, and neutron excess, respectively, and a_1, a_2, a_3, and a_4 are constants whose values are usually determined from mass and stability data. In elementary treatments of nuclear structure, Eq. (1) is usually derived using the liquid drop model of the nucleus, and the first three terms are physically interpreted as the volume, surface, and Coulomb energies. The last term, the so-called symmetry energy does not, however, have a ready interpretation in the liquid drop picture. In this article we give a simple alternative derivation of the Bethe–Weizsäcker formula, which is more in the spirit of the present self-consistent field view of the nucleus.[4,5]

[1] C. F. von Weizsäcker, Z. Physik **96**, 431 (1935).

[2] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 82 (1936).

[3] A. E. S. Green, Rev. Mod. Phys. **30**, 569 (1958).

[4] M. Baranger, *Cargese Lectures in Theoretical Physics*, M. Levy, Ed. (W. A. Benjamin, Inc., New York, 1963), Chap. 5, pp. 29–32.

[5] G. Brown, *Unified Theory of Nuclear Models and Forces* (John Wiley & Sons, Inc., New York, 1967).

I. THE LENNARD-JONES POTENTIAL AND BETHE-WEIZSÄCKER EQUATION

We begin with the assumption that the major interaction between pairs of nucleons in a nucleus is a charge independent Lennard-Jones potential of the form,

$$v=-g/r^{\alpha}+h/r^{\beta}, \quad (2)$$

where g and h are constants, $\beta>\alpha>1$, and r is the separation distance between nucleons. We also assume that neutron–proton pairs experience an additional "symmetry" attraction given by

$$v_s=-s/r^{\alpha}. \quad (3)$$

Summing over all nucleon pairs and including the Coulomb energy, the total potential energy of the nucleus becomes

$$V=\sum_{p=1}^{A(A-1)/2}-\frac{g}{r_p{}^{\alpha}}+\frac{h}{r_p{}^{\beta}}+\sum_{p=1}^{Z(Z-1)/2}\frac{e^2}{r_p}-\sum_{p=1}^{NZ}\frac{s}{r_p{}^{\alpha}}. \quad (4)$$

Considering the mean value theorem,

$$\sum_{p=}^{n}f(x_p)=nf\langle x\rangle_{\mathrm{av}}, \quad (5)$$

where $x_1<x_2<\cdots<x_n$, $f(x)$ is continuous on the interval $x_1<x<x_n$, and $\langle x\rangle_{\mathrm{av}}$ is some value on this interval, the various series in Eq. (4) may be

summed to yield

$$\sum_{p=}^{n} r_p^{-\gamma} = n/\langle r_{p\gamma}\rangle_{\mathrm{av}}{}^{\gamma}$$
$$= n/(r_\gamma A^{1/3})^{\gamma}. \tag{6}$$

In the last expression, we assume that any average separation moment $\langle r_{p\gamma}\rangle_{\mathrm{av}}{}^{\gamma}$ scales as $A^{1/3}$, where r_γ is a scaling distance which depends upon the power γ.

To determine the kinetic energy, we use the result from Fermi–Thomas theory that

$$N = 2(4\pi R^3/3)(p_m{}^3/6\pi^2\hbar^3), \tag{7}$$

represents the number of neutron states of both spins having momenta less than p_m. It is then easy to show that the total kinetic energy of this assembly of neutrons is given by[6]

$$T_n = \tfrac{3}{10}(\hbar^2/MR^2)(9\pi/4)^{2/3}N^{5/3}, \tag{8}$$

where M is an average nucleon mass. Using expressions of the same form for protons and $N=(A/2)[1+(D/A)]$ and $Z=(A/2)[1-(D/A)]$, neglecting terms in D^4/A^4 and higher, and assuming that $R=r_0A^{1/3}$, it follows that the kinetic energy of the nucleus is given by

$$T = T_0A + \tfrac{5}{9}T_0D^2/A, \tag{9}$$

where

$$T_0 = (9\pi/8)^{2/3}(3\hbar^2/10Mr_0{}^2). \tag{10}$$

From the experimental electron-scattering data,[7] it is estimated that the nuclear radius constant $r_0 \cong 1.12$ F, which yields $T_0 = 23.0$ MeV.

Using Eq. (5) to evaluate the summations in Eq. (4) and adding the kinetic energy as given by Eq. (9), we obtain for the total energy of the nucleus

$$E = T+V = T_0A - \frac{2A(A-1)g + A^2s}{4r_\alpha{}^\alpha A^{\alpha/3}} + \frac{A(A-1)h}{2r_\beta{}^\beta A^{\beta/3}} + \frac{Z(Z-1)e^2}{2r_1A^{1/3}} + \frac{D^2}{A}\left(\frac{5T_0}{9} + \frac{As}{4r_\alpha{}^\alpha A^{\alpha/3}}\right). \tag{11}$$

[6] A. E. S. Green, T. Sawada, and D. S. Saxon, *The Nuclear Independent Particle Model* (Academic Press, Inc., New York, to be published), Sec. 5.2.

[7] R. Hofstadter, Ann. Rev. Nucl. Sci. **7**, 231 (1957).

Letting $\alpha=3$, $\beta=4$, and neglecting unity as compared with A or Z, the total energy reduces to precisely the usual Bethe–Weizsäcker formula (Eq. 1).

II. DETERMINATION OF CONSTANTS

Identifying the mass formula constants with a set obtained from a best fit to the data,[3] we have

$$a_1 = (2g+s)/4r_3{}^3 - T_0 = 15.82,$$
$$a_2 = h/2r_4{}^4 = 17.90,$$
$$a_3 = e^2/2r_1 = 0.718,$$

and

$$a_4 = s/4r_3{}^3 + 5T_0/9 = 23.5, \tag{12}$$

where a's, g, h, and s are in million electron volts and all distances are in Fermis.

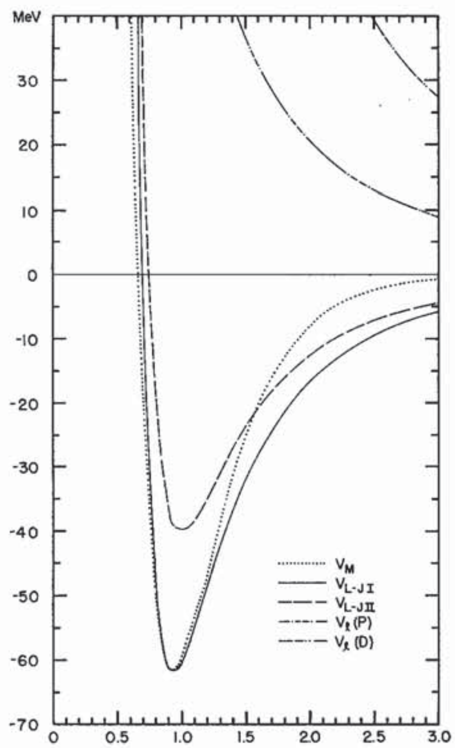

FIG. 1. N–N interaction potentials. V_M is ${}^1S_0$ n–p Morse potential of Darewych and Green. $V_{\mathrm{L-J\,I}}$ is a similar Lennard-Jones potential which matches V_M at minimum. $V_{\mathrm{L-J\,II}}$ is adjusted (through the kindness of T. Sawada and D. Sellin) to bind the ${}^1S_0$ state near zero energy. $V_l(P)$ and $V_l(D)$ are centrifugal potentials for P and D states of relative motion of an N–N pair.

From the last and first equalities, we have

$$s/4r_3{}^3=10.7,$$

and

$$g/2r_3{}^3=28.2. \tag{13}$$

We now appeal to studies of the N–N interaction to estimate g and h. Figure 1 shows the Lennard-Jones potential $V_{\mathrm{L-J\ II}}$, with $g=90.2$ and $h=67.6$ which gives a ${}^1S_0$ state near zero binding, as determined using the Abacus II code.[8] Figure 1 also shows the Lennard-Jones potential $V_{\mathrm{L-J\ I}}$, with $g=204$ and $h=143$ chosen to match the minimum of the Morse potential V_m for the ${}^1S_0$ np interaction as determined by Darewych and Green.[9] This Morse potential fits the experimental ${}^1S_0$ phase shifts very precisely from 0 to 350 MeV. The similarities of the two potentials suggest that by use of a judicious cutoff of the r^{-4} singularity and by minor adjustments in parameters, the Lennard-Jones potential could also provide a reasonable representation of N–N scattering data for the ${}^1S_0$ state. To deal with the ${}^3S_1$ state, we must determine the value of s.

Using the first set of values, we determine the radius parameters $r_3=1.17$ and $r_4=1.17$. These fall reasonably within the allowed limits 0 to $2r_0$. The Coulomb constant $r_1=1.00$ is roughly consistent with the estimate $r_1=5r_0/6$, which may be deduced from classical electrostatics. Accepting the value of r_3, we calculate $s=50.0$ for the constant associated with the additional symmetry interaction between np pairs.

III. DISCUSSION AND CONCLUSION

The physical origin of the symmetry interaction has been considered in many studies, particularly in connection with the explanation of the symmetry term in the shell and optical model potentials.[10] These studies suggest that the origin lies in the apparent spin dependence and l dependence of the N–N interaction in conjunction with the Pauli exclusion principle. On the average the l dependence may be roughly simulated by a Serber interaction of the form,

$$V=\tfrac{1}{2}[1+(-1)^l]V(r),$$

which vanishes in P, F, and other odd l states. The centrifugal interaction keeps nucleons outside the range of the nuclear interaction in D and G states. For S waves (symmetric in space) the nn and pp interactions (symmetric in isotopic spin) can only occur in ${}^1S$ states (antisymmetric in spin). However, np interactions which are mixtures of isotopic spin 0 and 1 can occur in ${}^1S_0$ and ${}^3S_1$ states. Using the statistical weights of these states, we find

$$\begin{aligned} v_s &= v_{np}-v_{nn} \\ &= (\tfrac{3}{4}{}^3v+\tfrac{1}{4}{}^1v)-{}^1v \\ &= \tfrac{3}{4}{}^3v-\tfrac{3}{4}{}^1v. \end{aligned} \tag{14}$$

The ${}^3S_1$ potential deduced from our estimated value of s is quite reasonable.

It might be remarked that a more realistic calculation of v_s for L-J ${}^3v$ and ${}^1v$ interactions would probably yield an r^{-4} repulsive term in addition to the r^{-3} attractive term. Such a symmetry interaction would lead directly to a so-called surface symmetry energy which arises in almost any derivation of the Bethe–Weizsäcker equation.[3] To be physically meaningful, however, we must then also include surface corrections to the kinetic energy, a refinement which would complicate our simple derivation. Accordingly, we have simply represented the symmetry interaction by an attractive term. Probably most of the added attraction is associated with the tensor force due to the π meson, although other N–N interaction components due to the ω, ρ, η, and other mesons also play a role.

In actuality recent meson theoretic descriptions of the N–N interaction,[11] the so-called One-Boson Exchange Potentials OBEP, have greatly clarified the nature of the N–N interaction. These studies reveal that the N–N interaction contains spin–spin, spin–orbit, tensor, and velocity-dependent interactions comparable in magnitude to the static central term. These interactions are very similar in structure to the relativistic interactions between two electrons. The application of such

[8] E. H. Auerbach, BNL-6562 (Brookhaven National Laboratory, Upton, New York, 1962) (adapted by D. L. Sellin).

[9] G. Darewych and A. E. S. Green, Phys. Rev. **164**, 1324 (1967).

[10] Reference 6, Sec. 2.3.

[11] A. E. S. Green, M. H. MacGregor, and R. Wilson, Eds., Proceedings of the International Conference on the Nucleon–Nucleon Interaction, Rev. Mod. Phys. **39**, 495 (1967).

realistic N–N interactions to the finite nucleus many-body problem remains at the very frontier of nuclear physics research.[12] To describe these studies would carry us beyond the scope of this article, which is intended primarily to provide an elementary derivation of the Bethe–Weizsäcker equation. Contrary to the elementary derivation based upon the liquid-drop model, the present derivation is within the conceptual spirit of current views of nuclear structure.

[12] M. Baranger, *Recent Progress in the Understanding of Finite Nuclei from the Two Nucleon Interaction, 1967 Varenna Lectures* (Carnegie-Mellon University, Pittsburgh, Pa., 1967).