UNIVERSITY OF CALIFORNIA, LOS ANGELES

The Electromagnetic Radiation from a Coaxial Structure

A thesis submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Physics

by

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Introduction

In the process of working toward an approximate solution of the problem of an antenna consisting of a coaxial line with the outside cylinder terminated and the inside cylinder extending a finite distance further, it was found that the exact solution of two other antenna problems could be obtained which would be useful for the approximate solution of the original problem. The two problems which allow an exact solution are an antenna consisting of a coaxial line with the outside cylinder terminated and the inside cylinder extending to infinity and an antenna consisting of a single half infinite cylinder (see Fig. 1). Inasmuch

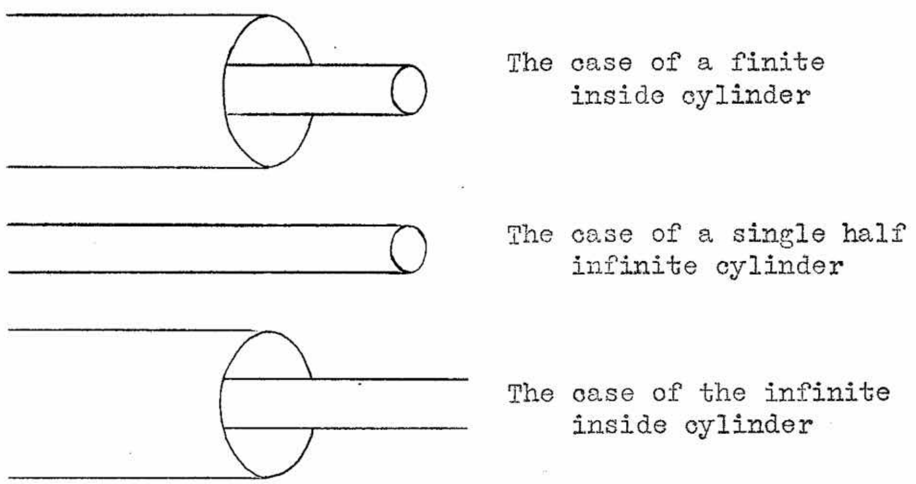

Fig. 1. The structures for the three problems considered.

as the three problems which resulted appear to be of about equal value, they are each presented on an equivalent and more or less independent basis.

The case of an infinite inside cylinder is stated and solved in Chapter I. The numerical results are presented in Chapter IV. Harold Levine solved this problem simultaneously and independently of the author; his results are presented in "Waveguide Handbook," (McGraw-Hill Co., New York, N. Y.), Edited by N. Marcuvitz, p. 208. A year before the publication of this book H. Levine sent the author a copy of his solution to the problem which checked in most particulars with the solution the author had previously found. A slight simplification is achieved by solving for E_z, the z component of the electric intensity, rather than H_ϕ, the ϕ component of the magnetic intensity, as Levine did. Levine's results contain an error where he failed to evaluate an integral properly for a particular choice of the parameter involved. The author made the identical mistake until the computations disclosed an improper discontinuity (see pp. 28a, 28b, 29). N. Marcuvitz in a letter to the author enclosing numerical results on this problem indicated that they were aware of this difficulty but had not resolved it. Apart from this one error the numerical values which can be compared seem to check to three significant figures.

The case of the single half infinite cylinder is stated and solved in Chapter II. No numerical results were obtained for this problem, since the idealized problem presented does not correspond to any actual physical situation. However, it is possible to interpret this problem physically if another parameter is introduced and the appropriate approximations are made. This was not

not done in the present thesis, since it would have been a digression from the primary object of finding a solution for the case of a finite inside cylinder.

The case of a finite inside cylinder is stated and solved approximately in Chapter III. The technique of obtaining this approximate solution appears crude; however, it was checked against an analytical method which allowed for an iterative process for successive approximations. Since each iteration required another integration, it was found to be practical only to the first approximation. The essential difference between the analytical type first approximation and that actually used was in the expression for Λ which is defined on page 73, equation (19.4), the analytical type first approximation yielding

$$\Lambda = -\frac{c'}{c} = \frac{\log\frac{b}{a}\, L^{+}(k)}{M^{+}(k)}\, e^{ikh}$$

which does not appear to be as satisfactory as the simple approximation used. The numerical results are presented in Chapter IV.

Chapter I

The case of an infinite inside cylinder

The problem of the infinite inside cylinder may be stated and solved exactly; therefore no reference is made in this chapter to the problem of the finite inside cylinder. The results of this chapter together with the results of Chapter II are used in Chapter III to obtain approximate solutions for the case of a finite inside cylinder. The numerical results for the present problem are contained in Chapter IV.

1. Statement of the problem.

The electromagnetic radiation from the following coaxial structure is to be studied. The structure is formed by two coaxial metallic cylinders of vanishing thickness, the inside cylinder of radius a extending from $-\infty$ to $+\infty$ along the z axis, and the outside cylinder of radius b extending from $-\infty$ to 0. Cylindrical coordinates ρ and z, and spherical coordinates r and θ are chosen as indicated in Fig. 2.

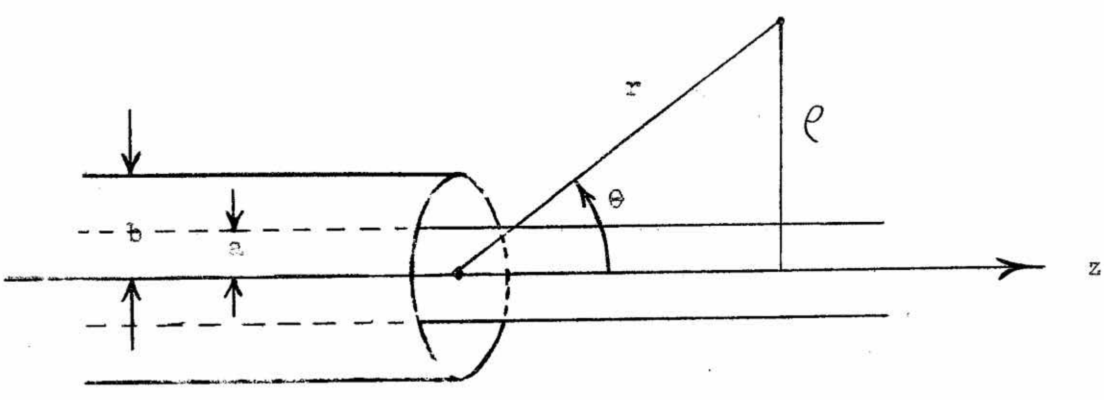

Fig. 2. The coaxial structure for an infinite inside cylinder

A source of electromagnetic waves is assumed to exist between the cylinders, i.e., the coaxial region, for $z \to -\infty$. The problem then becomes one of solving Maxwell's equations for vacuum (or air) with boundaries of an ideal metal, i.e., with infinite conductivity. It will be assumed that the time variation for all the field components is harmonic with angular frequency ω. It will also be assumed that $\omega < \pi c/(b - a)$ where c is the velocity of light; so that only the TEM or principal mode is propagated in the coaxial region.[1] The form of the solution in the coaxial region for $z \to -\infty$ is then

$$E_\rho = \frac{1}{\eta \rho} (Ae^{ikz} - Be^{ikz}) , \qquad (1.1)$$

$$H_\phi = \frac{1}{\rho} (Ae^{ikz} + Be^{-ikz}) , \qquad (1.2)$$

with

$$E_z, H_z, E_\phi, H_\rho = 0, \qquad (1.3)$$

where E designates the electric intensity, H designates the magnetic intensity, and the subscripts indicate the various components. The remaining symbols are defined as follows: A and B are as yet undetermined constants; $k = \omega\sqrt{\mu_0 \epsilon_0}$ is the propagation constant for plane homogeneous waves in free space where ϵ_0 is the specific electric inductive capacity and μ_0 is the specific magnetic inductive capacity; and $\eta = \sqrt{\epsilon_0/\mu_0}$ is the so-called intrinsic

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[1] J. A. Stratton, "Electromagnetic Theory" (McGraw-Hill, New York, 1941), pp. 545-51, Ref. (1).

admittance. It is obvious from the geometry and from the form of the solution in the coaxial region for $z \rightarrow -\infty$ that the fields must remain axially symmetric everywhere. Under these conditions it can be shown that Maxwell's equations reduce to the following:[2]

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\partial^2}{\partial z^2} + k^2\right)E_z = 0\ , \tag{1.4}$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)E_\rho = \frac{\partial^2 E_z}{\partial\rho\,\partial z} \tag{1.5}$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)H_\phi = ik\eta\frac{\partial E_z}{\partial\rho}\ , \tag{1.6}$$

with E_ϕ, H_z, $H_\rho = 0$ everywhere, subject to the boundary conditions:

$$E_z = 0 \tag{1.7}$$

and

$$\left(\frac{1}{\rho} + \frac{\partial}{\partial\rho}\right)H_\phi = 0 \tag{1.8}$$

for $\rho = b$, $z \leq 0$; and $\rho = a$, $-\infty < z < +\infty$. In addition it will be necessary to apply the radiation condition outside the coaxial region,

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[2] See Appendix A for this derivation.

$$E_z \,,\, E_\rho \,,\, H_\phi \propto \frac{e^{ik|\vec{r}|}}{|\vec{r}|} \tag{1.9}$$

as $|\vec{r}| = \sqrt{\rho^2 + z^2} \rightarrow \infty$. Once E_z has been found, equations (1.5) and (1.6) may be used to determine E_ρ and H_ϕ; and all of the characteristics of the electromagnetic field may then be determined.

2. Derivation of the integral equation.[3]

Consider the Green's function $K(\vec{r},\vec{r}')$ defined by

$$(\nabla^2 + k^2)\, K(\vec{r},\vec{r}') = -\,\delta(\vec{r} - \vec{r}') \tag{2.1}$$

where the del squared operator for the present case of axial symmetry is given in equation (1.4) and $\delta(\vec{r} - \vec{r}')$ is the usual Dirac delta function; the boundary condition

$$K(\vec{r},\vec{r}') = 0 \text{ for } \rho = a \,,\, -\infty < z < +\infty \;; \tag{2.2}$$

and the radiation condition

$$K(\vec{r},\vec{r}') \propto \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \text{ for } |\vec{r} - \vec{r}'| \rightarrow \infty, \tag{2.3}$$

ρ and $\rho' > a$ where $\vec{r}$ is the position vector of an observer and $\vec{r}'$ is the position vector of an oscillating point source. Applying Green's second identity to the two functions $E_z(\vec{r})$ and $K(\vec{r},\vec{r}')$, we obtain

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[3] H. Levine and J. Schwinger, Phys. Rev. 73, 389(1948), Ref. (2).

$$\int_{\text{Volume}} \left\{ K\,(\vec{r},\vec{r}\,')\,(\nabla'^2 + k^2)\,E_z\,(\vec{r}\,') - E_z(\vec{r}\,')\,(\nabla'^2 + k^2)\,K\,(\vec{r},\vec{r}\,') \right\} d\tau'$$

$$= \int_{\text{Surfaces}} \left\{ K\,(\vec{r},\vec{r}\,')\,\frac{\partial E_z(\vec{r}\,')}{\partial n'} - E_z\,(\vec{r}\,')\,\frac{\partial K}{\partial n'}\,(\vec{r}\,',\vec{r}) \right\} da'. \qquad (2.4)$$

Using (1.4) and (2.1), the volume integral on the left reduces simply to $E_z(\vec{r})$. For the surface integral on the right the surfaces are chosen as follows (see Fig. 3): S_1 is the surface of the inside cylinder; S_2 is the surface in the coaxial region for $z \to -\infty$; S_3 is the inside surface of the outside cylinder; S_4 is the outside surface of the outside cylinder; and S_5 is the remainder of the surface of the sphere at infinity.

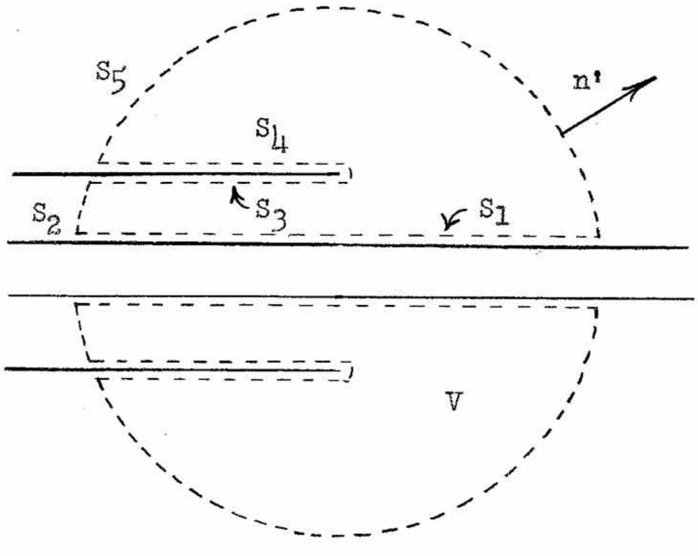

Fig. 3. Region of application of Green's second identity.

The behavior of E_z and K on these surfaces is as follows: on S_1, $E_z = 0$ from (1.6) and $K = 0$ from (2.2); on S_2, E_z and $\frac{\partial E_z}{\partial n'} = 0$ from (1.3); on S_3, $E_z = 0$ from (1.7); on S_4, $E_z = 0$ from (1.7); and on S_5, E_z, $\frac{\partial E_z}{\partial n'} \propto \frac{e^{ik|\vec{r}'|}}{|\vec{r}'|}$ from (1.9) and K, $\frac{\partial K}{\partial n'} \propto \frac{e^{ik|\vec{r}'|}}{|\vec{r}'|}$ from (2.3) where k is assumed to have an arbitrarily small imaginary part, $k = \beta + i\alpha$. Green's second identity (2.4) then reduces to

$$E_z(\vec{r}) = \int_{S_3} K(\vec{r}',\vec{r}) \frac{\partial E_z(\vec{r}')}{\partial \rho'} da' - \int_{S_4} K(\vec{r}',\vec{r}) \frac{\partial E_z(\vec{r}')}{\partial \rho'} da' . \quad (2.5)$$

Let

$$\phi(z') = - b \frac{\partial E_z(\rho', z')}{\partial \rho'} \Bigg|_{\rho' = b - 0}^{\rho' = b + 0} \quad (2.6)$$

where $\rho' = b + 0$ means that the expression is evaluated on the outside surface of the cylinder, and $\rho' = b - 0$ means the expression is evaluated on the inside surface of the cylinder. Equation (2.5) then becomes

$$E_z(\rho, z) = \int_{-\infty}^{0} K(b, \rho, z', z) \phi(z') dz' \quad (2.7)$$

where we note that

$$K(\vec{r}', \vec{r}) = \frac{K(\rho', \rho, z', z)}{2\pi}$$

as a result of the axial symmetry. In order to obtain the integral equation the boundary conditions on $E_z(\rho, z)$ are used, i.e.,

$$E_z(b,z) = \begin{cases} E_z(b,z) & \text{for } z > 0 \\ 0 & \text{for } z \leq 0 . \end{cases}$$

Similarly using definition (2.6), we find

$$\phi(z') = \begin{cases} \phi(z') & \text{for } z' \leq 0 \\ 0 & \text{for } z' > 0 . \end{cases}$$

It can also be shown[4] that $K(b, \rho, z', z) = K(b, \rho, z - z')$. The integral equation,

$$E_z(b,z) = \int_{-\infty}^{\infty} K(b, b, z - z') \, \phi(z') \, dz' , \qquad (2.8)$$

resembles the Wiener-Hopf type of homogeneous integral equation[5] which can be solved by the application of a Fourier transform method.[6] In particular the unknown function, $\phi(z')$, can be found from the solution of (2.8) and substituted into (2.7) to yield the desired field variable $E_z(\rho, z)$.

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[4] See Appendix B

[5] E. C. Titchmarsh, "Introduction to the Theory of Fourier Integrals" (Oxford University Press, London, 1937), Chapter IV, p. 339, Ref. (3).

[6] Levine and Schwinger, op. cit., p. 393, Ref. (2).

3. Derivation of the transform equation[7]

The Fourier transforms of E_z (b,z) and ϕ (z) are defined as follows:

$$\mathcal{E}_z(\zeta) = \int_{-\infty}^{\infty} E_z(b,z)\, e^{-i\zeta z}\, dz = \int_{-o}^{\infty} E_z(b,z)\, e^{-i\zeta z}\, dz\,, \quad (3.1)$$

and

$$\Phi(\zeta) = \int_{-\infty}^{\infty} \phi(z) e^{-i\zeta z} dz = -\int_{-\infty}^{o} b\, \frac{\partial E_z(\rho,z)}{\partial \rho} \Bigg|_{\rho=b-o}^{\rho=b+o} e^{-i\zeta z} dz\,. \quad (3.2)$$

Multiplying both sides of equation (2.8) by $e^{-i\zeta z}$ and integrating with respect to z from $z = -\infty$ to $+\infty$, and assuming for the moment that it is possible to change the order of integration on the right, equation (2.8) becomes

$$\int_{-\infty}^{\infty} E_z(b,z)\, e^{-i\zeta z}\, dz =$$

$$\int_{-\infty}^{\infty} K(b,b,z-z')e^{-i(z-z')\zeta} d(z-z') \int_{-\infty}^{\infty} \phi(z')e^{-iz'\zeta} dz'. \quad (3.4)$$

Substituting the definition of the transforms into (3.4) the transform equation is obtained,

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[7] Loc. cit., Ref. (2).

$$\mathcal{E}_z(b,\zeta) = \mathcal{K}(b,\, b,\, \zeta)\, \Phi(\zeta)\ ;$$

or merely

$$\mathcal{E}_z(\zeta) = \mathcal{K}(\zeta)\, \Phi(\zeta) \tag{3.3}$$

where[8]

$$\mathcal{K}(\rho_>, \rho_<, \zeta) = \frac{H_0^{(1)}(\gamma\rho_>)}{H_0^{(1)}(\gamma a)}\, Z_0(\gamma\rho_<) \tag{3.4}$$

where

$$Z_0(\gamma\rho_<) = \frac{\pi}{2}\left\{J_0(\gamma a)\, N_0(\gamma\rho_<) - N_0(\gamma a)\, J_0(\gamma\rho_<)\right\}, \tag{3.4a}$$

and $\gamma = \sqrt{k^2 - \zeta^2}$. The phase of the radical is chosen so that for ζ on the real axis (k considered real) the phase is 0 for $|\zeta| < k$ and $\frac{\pi}{2}$ for $|\zeta| > k$. According to the convolution theorem, the above application will only be valid in the common region of analyticity of the three transforms.

We now verify that such a region exists. The Green's function transform is analytic in the strip $|\mathrm{Im}\,\zeta| < \alpha$ where[9] $\mathrm{Im}\,k = \alpha$. In order to determine the region in which $\Phi(\zeta)$ is defined, substitute the asymptotic form of the integrand for $z \to -\infty$ into the definition, (3.2). Using the assumed asymptotic form (1.9) of E_z outside the coaxial region and noting that $E_z \equiv 0$ inside the

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8 See Appendix B.

9 Loc. cit.

coaxial region, the integrand is seen to vary, putting $\zeta = \xi + i\eta$, as

$$\frac{e^{-ikz}}{z} e^{-i\zeta z} \sim e^{(\alpha+\eta)z} \text{ for } z \rightarrow -\infty .$$

Then

$$|\bar{\Phi}(\zeta)| \leq \int_{-\infty}^{0} |\phi(z)| \, e^{\eta z} \, dz$$

is bounded for Im $\zeta < -\alpha$, since $|\phi(z)|$ remains finite over the range of integration. Therefore $\bar{\Phi}(\zeta)$ is analytic in the upper half plane, Im $\zeta > -\alpha$. In order to determine the region in which $\mathcal{E}_z(\zeta)$ is defined, substitute the asymptotic form (1.9), for $z \rightarrow +\infty$, into the definition, (3.1). The integrand is seen to vary as

$$\frac{e^{ikz}}{z} e^{-i\zeta z} \sim e^{-(\alpha-\eta)z} \text{ for } z \rightarrow +\infty .$$

Then

$$|\mathcal{E}_z(\zeta)| \leq \int_{0}^{\infty} |E_z(b,z)| \, e^{\eta z} \, dz$$

is bounded for Im $\zeta < \alpha$, since $|E_z(b,z)|$ remains finite over the range of integration except at the origin where it is integrable.[10] Therefore $\mathcal{E}_z(\zeta)$ is analytic in the lower half plane Im $\zeta < \alpha$. These results are summarized in Fig. 4. Therefore

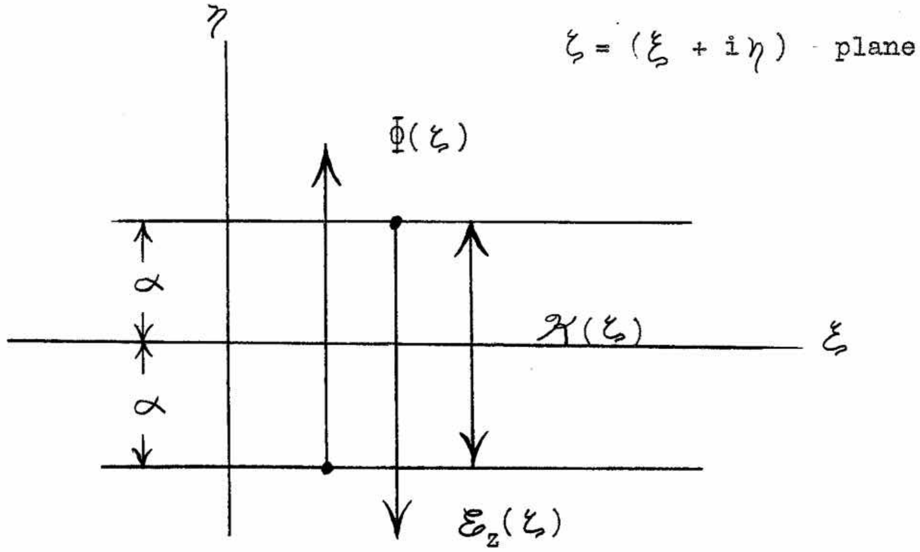

Fig. 4. Regions of analyticity of the transforms.

the common region of analyticity for which the transform equation is valid is the strip $|\text{Im}\,\zeta| < \alpha$.

4. <u>Procedure of solution of the transform equation</u>[11]

The next step is to write $\mathcal{K}(\zeta)$ as a ratio of two functions,

$$\mathcal{K}(\zeta) = M^+(\zeta) \,/\, M^-(\zeta) \,, \tag{4.1}$$

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[10] Cf. post, Section 7, p.

[11] Levine and Schwinger, op. cit., p. 395, Ref. (2).

where $M^+(\zeta)$ is analytic and has no zeros in the upper half plane, $\mathrm{Im}\,\zeta > -\alpha$, and $M^-(\zeta)$ is analytic and has no zeros in the lower half plane, $\mathrm{Im}\,\zeta < \alpha$. The transform equation then becomes

$$M^+(\zeta)\,\Phi(\zeta) = M^-(\zeta)\,\mathcal{E}_z(\zeta)\,. \tag{4.2}$$

The left side is analytic in the upper half plane, and the right side is analytic in the lower half plane. Since the two sides of the equation have a region in common (the strip $|\mathrm{Im}\,\zeta| < \alpha$) for which they are simultaneously analytic, the right side must be the analytic continuation of the left side. Together they represent a function, $f(\zeta)$, analytic in the finite ζ plane:

$$f(\zeta) \equiv \begin{cases} M^-(\zeta)\,\mathcal{E}_z(\zeta) \text{ for } \mathrm{Im}\,\zeta < \alpha \\ \text{Either expression for } |\mathrm{Im}\,\zeta| < \alpha. \\ M^+(\zeta)\,\Phi(\zeta) \text{ for } \mathrm{Im}\,\zeta > -\alpha \end{cases} \tag{4.3}$$

If this function, $f(\zeta)$, is analytic for all points in the finite ζ plane and $\lim_{|\zeta|\to\infty} |f(\zeta)|$ remains bounded, then according to Liouville's theorem,[12] the function must be identically a constant, $f(\zeta) \equiv c$. This gives the results:

$$\Phi(\zeta) = c \,/\, M^+(\zeta) \tag{4.4}$$

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[12] E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis' (Am. Ed., Macmillan Co., New York, 1948), Chapter V, p. 105, Ref. (4).

and

$$\mathcal{E}_z(\zeta) = c\,/\,M^-(\zeta)\,. \tag{4.5}$$

Then from (3.3)

$$\mathcal{E}_z(\rho,\zeta) = c\,\frac{\mathcal{K}(b,\rho,\zeta)}{M^+(\zeta)}\,. \tag{4.6}$$

Taking the inverse transform,

$$E_z(\rho,z) = \frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{\mathcal{K}(b,\rho,\zeta)}{M^+(\zeta)}\,e^{i\zeta z}\,d\zeta\,.$$

In order to evaluate $M^+(\zeta)$ and $M^-(\zeta)$, the general method of procedure is to apply Cauchy's integral theorem. Consider $\mathcal{K}(\zeta)$ in the region in which the function is regular. Apply Cauchy's integral formula to $\log \mathcal{K}(\zeta)$:

$$\log \mathcal{K}(\zeta) = \frac{1}{2\pi i}\oint\frac{\log \mathcal{K}(t)}{t-\zeta}\,dt\,. \tag{4.7}$$

Since $\mathcal{K}(\zeta)$ is regular in the strip $|\mathrm{Im}\,\zeta| < \alpha$ and has no zeros in this strip, $\log \mathcal{K}(\zeta)$ is regular in this strip (where the principal value of the logarithm is understood). Consider a rectangular contour confined to this strip with ends displaced to infinity (see Fig. 5.). In this strip $0 \leq \arg\sqrt{k^2 - t^2} \leq \frac{\pi}{2}$;

and [13] for $|t| \to \infty$, $\mathcal{K}(t) \sim 1/|t|$. The integrand for $|t| \to \infty$ in this strip becomes

$$\frac{\log \mathcal{K}(t)}{t - \zeta} \sim - \frac{\log |t|}{t} \to 0 .$$

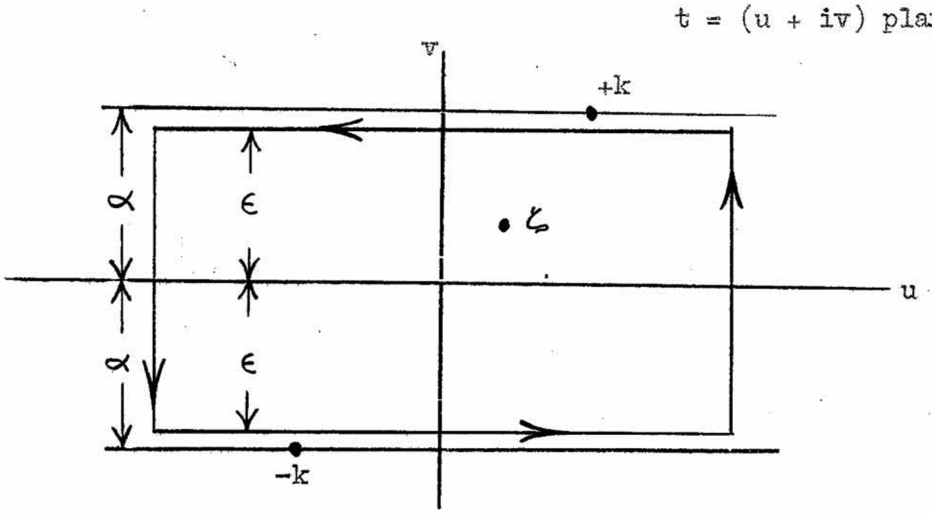

Fig. 5. Integration contour for $M^{+}(\zeta)$ and $M^{-}(\zeta)$.

Thus the contributions from the ends become negligible. Equation (4.7) then becomes

$$\log \mathcal{K}(\zeta) = \frac{1}{2\pi i} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{\log \mathcal{K}(t)}{t - \zeta}\, dt - \frac{1}{2\pi i} \int_{-\infty + i\epsilon}^{\infty + i\epsilon} \frac{\log \mathcal{K}(t)}{t - \zeta}\, dt$$

If the first integral converges, it is an analytic function for ζ anywhere in the upper half plane, $\mathrm{Im}\ \zeta > -\epsilon$; since the integration is restricted to a strip in which the integrand is regular. The first integral then defines a function, $\log M^{+}(\zeta)$,

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[13] See Appendix B.

analytic in the upper half plane, Im $\zeta > -\alpha$ (where it is permissible to let $\alpha - \epsilon \rightarrow 0$). Similarly the second integral defines a function, $\log M^-(\zeta)$, analytic in the lower half plane Im $\zeta < \alpha$. Taking anti-logarithms equation (4.1) is obtained. In the limit as $\alpha \rightarrow 0$, we have

$$M^+(\zeta) = 1 / M^-(\zeta) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log \mathcal{K}(t)}{t - \zeta}\,dt\right]. \qquad (4.9)$$

5. First evaluation[14] of $M^+(\zeta)$

In this and the next section two expressions for $M^+(\zeta)$ are derived. The reason for obtaining two expressions which must be equivalent is merely to be able to choose the more convenient expression for numerical computations. The interval of integration in (4.9) is broken up as follows:

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{-k} + \int_{k}^{\infty} + \int_{-k}^{o} + \int_{o}^{k}.$$

Combining the integrals for $t < o$ with the corresponding integrals for $t > o$ and introducing the proper phases of the square root, $\sqrt{k^2 - t^2}$, $M^+(\zeta)$ becomes

$$M^+(\zeta) = \exp\left[\frac{\zeta}{\pi i}\int_{o}^{k}\frac{\log \mathcal{K}(\gamma b)}{t^2 - \zeta^2}\,dt + \frac{\zeta}{\pi i}\int_{k}^{\infty}\frac{\log \mathcal{K}(i\gamma' b)}{t^2 - \zeta^2}\,dt\right] \quad (5.1)$$

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[14] Levine and Schwinger, op. cit. p. 396, Ref. (2).

where $\mathcal{H}(\gamma b) \equiv \mathcal{H}(t)$ is given by (3.4) and [15]

$$\mathcal{H}(i\gamma' b) \equiv \mathcal{H}(t) =$$

$$\frac{K_0(\gamma' b)}{K_0(\gamma' a)} \left\{ I_0(\gamma' b)\, K_0(\gamma' a) - I_0(\gamma' a)\, K_0(\gamma' b) \right\} \qquad (5.2)$$

where $\gamma' = \sqrt{t^2 - k^2}$. The combination of the two integrals for $t < -k$ and $t > k$ insures the convergence of the resulting integral from k to ∞, the original integrals being individually nonconvergent. When ζ is real, the singular integral in (5.1) may be evaluated as the principal value plus πi times the residue of the integrand at the pole $t = \zeta$. The plus sign is chosen since the contour is indented below the real axis. Let $x = \gamma$ for the first integral, and $x = \gamma'$ for the second integral. Then for ζ real and $\zeta < k$

$$M^+(\zeta) = \sqrt{\mathcal{H}(\zeta)}\ \exp \left[\frac{i\zeta}{\pi}\ P \int_0^k \frac{x \log \mathcal{H}(bx)}{[x^2 - (k^2 - \zeta^2)]\sqrt{k^2 - x^2}}\, dx \right.$$

$$\left. + \frac{i\zeta}{\pi} \int_0^\infty \frac{x \log \mathcal{H}(ibx)}{[x^2 + (k^2 - \zeta^2)]\sqrt{k^2 + x^2}}\, dx \right] \qquad (5.3)$$

where P designates the principal value. Writing $M^+(\zeta)$ in polar form for $\operatorname{Im} \zeta = 0$, $|\zeta| \leq k$, we get

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15 See Appendix B.

(5.4)

$$|M^+(\zeta)| = \sqrt{|\mathcal{H}(\zeta)|}$$

$$\times \exp\left[-\frac{\zeta}{\pi} P \int_0^k \frac{x\left\{\tan^{-1}\left(-\frac{J_0(bx)}{N_0(bx)}\right) - \tan^{-1}\left(-\frac{J_0(ax)}{N_0(ax)}\right)\right\}}{[x^2 - (k^2 - \zeta^2)]\sqrt{k^2 - x^2}}\, dr\right]$$

and

$$\arg\,(M^+(\zeta)) = \tfrac{1}{2}\left\{\tan^{-1}\left(-\frac{J_0(\gamma b)}{N_0(\gamma b)}\right) - \tan^{-1}\left(-\frac{J_0(\gamma a)}{N_0(\gamma a)}\right)\right\}$$

$$+ \frac{\zeta}{\pi} P \int_0^k \frac{x \log |\mathcal{H}(bx)|}{[x^2 - (k^2 - \zeta^2)]\sqrt{k^2 - x^2}}\, dx \qquad (5.5)$$

$$+ \frac{\zeta}{\pi} \int_0^\infty \frac{x \log \mathcal{H}(ibx)}{[x^2 + (k^2 - \zeta^2)]\sqrt{k^2 + x^2}}\, dx\ .$$

Evaluating (5.4) and (5.5) for $\zeta = k$, we get

$$|M^+(k)| =$$

$$\sqrt{\log b/a}\, \exp\left[-\frac{k}{\pi} \int_0^k \frac{\left\{\tan^{-1}\left(-\frac{J_0(bx)}{N_0(bx)}\right) - \tan\left(-\frac{J_0(ax)}{N_0(ax)}\right)\right\}}{x\sqrt{k^2 - x^2}}\, dx\right] \qquad (5.6)$$

and

$$\arg(M^+(k)) = \frac{k}{\pi}\int_0^k \frac{\log|\mathcal{X}(bx)|}{x\sqrt{k^2-x^2}}\,dx + \frac{k}{\pi}\int_0^\infty \frac{\log\mathcal{X}(ibx)}{x\sqrt{k^2+x^2}}\,dx \,. \quad (5.7)$$

6. Second evaluation[16] of $M^+(\zeta)$

Consider the logarithmic derivative of $M^+(\zeta)$ as given by (4.9),

$$\frac{d}{d\zeta}\log M^+(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log\mathcal{X}(t)}{(t-\zeta)^2}\,dt \,. \qquad (6.1)$$

In order to avoid questions of convergence take the limits of integration as $-T$ to $+T$, and later let $T \rightarrow \infty$. The path of integration along the real axis may be broken up as follows: let Γ_1 be a path from $t = -T$ above the branch cut to $t = -k$ and back to $t = -T$ below the branch cut; and let Γ_2 be a path from $t = -T$ to $t = -k$ below the branch cut and from $t = -k$ to $t = +T$ along the real axis (see Fig. 6).

[16] Levine and Schwinger, op. cit., p. 396, Ref. (2).

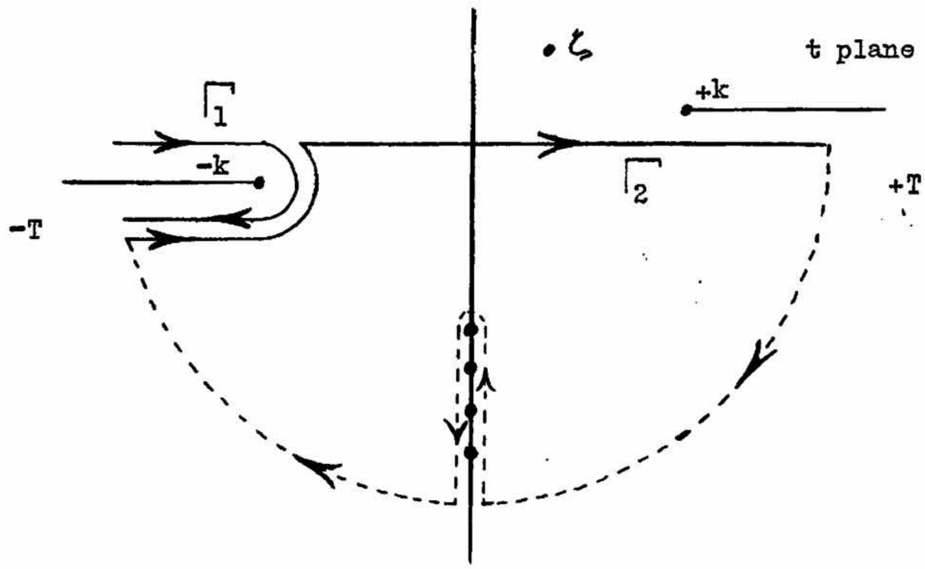

Fig. 6. Integration contours for the second evaluation of $M^+(\zeta)$.

Then we have

$$\int_{-T}^{+T} = \int_{\Gamma_1} + \int_{\Gamma_2} . \qquad (6.2)$$

Consider the integration over the contour Γ_1. Since $Z_0(\gamma b)$ is regular in the region of the branch cut, we have, upon introducing the proper phases of $\sqrt{k^2 - t^2}$ and combining the two integrals for the path above and below the cut,

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{\log \mathcal{X}(t)}{(t-\zeta)^2}\, dt = - \frac{1}{2\pi i} \int_k^T \log \left(\frac{\dfrac{\pi I_0(\gamma' b)}{K_0(\gamma' b)} - i}{\dfrac{\pi I_0(\gamma' a)}{K_0(\gamma' a)} - i} \right) \frac{dt}{(t+\zeta)^2} . \quad (6.3)$$

The asymptotic form of the integrand is

$$\frac{\log\, e^{2\sqrt{t^2-k^2}(b-a}}{(t+\zeta)^2} \qquad (6.4)$$

which[17] varies as $\frac{1}{t}$ for $t \to \infty$; therefore the integral diverges logarithmically for $T \to \infty$. In order to isolate this singularity, add

$$-\frac{\log\, e^{-2\sqrt{t^2-k^2}(b-a)}}{(t+\zeta)^2} - \frac{2\sqrt{t^2-k^2}\,(b-a)}{(t+\zeta)^2}$$

a quantity which is zero, to the integrand, where we note that

$$-\lim_{T\to\infty} \frac{1}{2\pi i}\int_k^T \frac{2\sqrt{t^2-k^2}\,(b-a)}{(t+\zeta)^2}\,dt = \frac{i(b-a)}{\pi}\lim_{T\to\infty} \log\frac{2T}{k}$$

$$-\frac{i(b-a)}{\pi}\left\{1 + \frac{2\,\zeta}{\sqrt{k^2-\zeta^2}}\tan^{-1}\frac{\sqrt{k^2-\zeta^2}}{k+\zeta}\right\}. \qquad (6.5)$$

Let

$$F(\zeta) = \exp\left[\frac{1}{2\pi i}\int_k^\infty \log\left\{\left(\frac{\frac{\pi I_0(\gamma' b)}{K_0(\gamma' b)} - i}{\frac{\pi I_0(\gamma' a)}{K_0(\gamma' a)} - i}\right) e^{-2\gamma'(b-a)}\right\}\frac{dt}{t+\zeta}\right]. \qquad (6.6)$$

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[17] See Appendix B.

Then the contribution over Γ_1 becomes

$$\frac{d}{d\zeta} \log F(\zeta) - \frac{i(b-a)}{\pi} \left\{ 1 + \frac{2\zeta}{\sqrt{k^2 - \zeta^2}} \tan^{-1} \frac{\sqrt{k^2 - \zeta^2}}{k + \zeta} \right\}$$

$$+ \frac{i(b-a)}{\pi} \lim_{T \to \infty} \log \frac{2T}{k} . \qquad (6.7)$$

In order to evaluate the integral over Γ_2, we close the path with circular arcs at infinity and two paths passing on the two sides of a branch cut taken along the negative imaginary axis thru the zeros of $Z_0(\gamma b) = 0$ occurring at $t = -i\sqrt{\gamma_n^2 - k^2}$ (see Fig. 6). Since this contour incloses no singularities, we have

$$\int_{\Gamma_2} = -\int_{\text{arcs}} - \int_{\text{branch cut}} . \qquad (6.8)$$

First evaluating the integral over the arcs for $|t| \to \infty$ in the lower half plane where $\sqrt{k^2 - t^2} \to it$, we have[18]

$$\mathcal{H}(t) \sim -\frac{1}{2tb} \left(e^{-2t(b-a)} - 1 \right) .$$

In the fourth quadrant the integrand of (6.1) then becomes

$$\frac{\log \frac{1}{2}tb}{(t - \zeta)^2}$$

[18] Loc. cit.

which approaches zero in such a way that there is no contribution from the integral over the arc in the fourth quadrant. In the third quadrant the integrand of (6.1) becomes

$$-\frac{2(b-a)}{t} .$$

Then in the limit as $T \rightarrow \infty$, the contribution over the arcs becomes

$$\frac{1}{2\pi i}\int_{(-\frac{\pi}{2})}^{(-\pi)}\left[-\frac{2(b-a)}{t}\right] dt = \frac{b-a}{2} . \qquad (6.9)$$

Second, we evaluate the integral over the branch cut. Since the asymptotic form of the roots is given by

$$\gamma_n = \frac{n\pi}{b-a} + O(\frac{1}{n}) , \qquad (6.10)$$

the number of roots in the interval, 0 to $-T$ on the imaginary axis is given by

$$n \approx \frac{b-a}{\pi}\,\gamma_n = \left[\frac{b-a}{\pi}\,T\right] \equiv N$$

Where $\left[\frac{b-a}{\pi}\,T\right]$ signifies the largest integer less than $\frac{b-a}{\pi}\,T$.
The functions $H_0^{(1)}(\gamma b) \,/\, H_0^{(1)}(\gamma a)$ and $\dfrac{Z_0(\gamma b)}{\prod_{n=1}^{N}\left(t + i\sqrt{\gamma_n^2 - k^2}\right)}$

are regular in the region of the negative imaginary axis and therefore do not contribute to the integral. The remaining integral to be calculated is

$$\frac{1}{2\pi i}\int_{\text{branch cut}} \log \prod_{n=1}^{N}\left(t + i\sqrt{\gamma_n^{\,2}-k^2}\right)\frac{dt}{(t-\zeta)^2} \qquad (6.11)$$

where the phase of $(t + i\sqrt{\gamma_n^{\,2}-k^2})$ must be chosen 2π degrees greater for the integral over the left side of the cut as compared with the integral over the right side. Thus (6.11) becomes

$$\sum_{n=1}^{N}\int_{-i\sqrt{\gamma_n^{\,2}-k^2}}^{-iT}\frac{dt}{(t-\zeta)^2} = \sum_{n=1}^{N}\left(\frac{1}{\zeta + iT} - \frac{1}{\zeta + i\sqrt{\gamma_n^{\,2}-k^2}}\right) . \qquad (6.12)$$

But since $\lim_{m\to\infty}\left[\sum_{n=1}^{m}\frac{1}{n} - \log m\right] = \log \beta = c = 0.5772$ where c is Euler's constant,[19] we have from (6.12)

$$\lim_{T\to\infty}\sum_{n=1}^{N}\left[\frac{1}{\zeta - iT} - \left\{\frac{i(b-a)}{n\pi} + \frac{1}{\zeta + i\sqrt{\gamma_n^{\,2}-k^2}}\right\} + \frac{i(b-a)}{n\pi}\right]$$

$$= -\frac{i(b-a)}{\pi} - \sum_{n=1}^{\infty}\left\{\frac{i(b-a)}{n\pi} + \frac{1}{\zeta + i\sqrt{\gamma_n^{\,2}-k^2}}\right\}$$

$$+ \frac{i(b-a)}{\pi}\lim_{T\to\infty}\log\frac{(b-a)}{\pi}T\beta . \qquad (6.13)$$

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19 Whittacker and Watson, op. cit., p. 235, Ref. (4).

Collecting results (6.13), (6.9), and (6.7) and combining them according to (6.8) and (6.2), we get

$$\frac{d}{d\zeta}\log M^{+}(\zeta) = \frac{d}{d\zeta}\log F(\zeta) - \frac{i(b-a)}{\pi}\log\frac{(b-a)k\beta}{2\pi i}$$

$$-\frac{2i(b-a)\zeta}{\pi\sqrt{k^2-\zeta^2}}\tan^{-1}\frac{\sqrt{k^2-\zeta^2}}{k+\zeta}+\sum_{n=1}^{\infty}\left(\frac{i(b-a)}{n\pi}+\frac{1}{\zeta+i\sqrt{\gamma_n^{\,2}-k^2}}\right) \quad (6.14)$$

where the two diverging terms conveniently cancel each other. It is now necessary to integrate (6.14) in order to obtain $M^{+}(\zeta)$. Using the relations

$$\exp\left[\int\sum_{n=1}^{\infty}\left(\frac{i(b-a)}{n\pi}+\frac{1}{\zeta+i\sqrt{\gamma_n^{\,2}-k^2}}\right)d\zeta\right] =$$

$$\text{const.}\ \log\left\{\prod_{n=1}^{\infty}(i\gamma_n)\right\}\prod_{n=1}^{\infty}\left(\sqrt{1-\frac{k^2}{\gamma_n^{\,2}}}-\frac{i\zeta}{\gamma_n}\right)e^{\frac{i(b-a)\zeta}{n\pi}}$$

and

$$\int-\frac{\zeta}{\sqrt{k^2-\zeta^2}}\tan^{-1}\frac{\sqrt{k^2-\zeta^2}}{k+\zeta}\,d\zeta = -\frac{\zeta}{2}+\sqrt{k^2-\zeta^2}\tan^{-1}\frac{\sqrt{k^2-\zeta^2}}{k+\zeta}+\text{const.}\,,$$

we obtain

$$M^+(\zeta) = CF(\zeta) \prod_{n=1}^{\infty} \left(\sqrt{1 - \frac{k^2}{\gamma_n^{\,2}}} - \frac{i\zeta}{\gamma_n} \right) e^{\frac{i(b-a)\zeta}{n\pi}}$$

$$\times \exp \left[\frac{i(b-a)}{\pi} \zeta \left(\log \frac{2\pi i}{k\beta(b-a)} + 1 \right) + \frac{2i(b-a)}{\pi} \sqrt{k^2 - \zeta^2} \tan^{-1} \frac{\sqrt{k^2-\zeta^2}}{k+\zeta} \right] \qquad (6.15)$$

where C is a constant of integration.

In order to determine C, we note that in the strip, $|\mathrm{Im}\,\zeta| < \alpha$,

$$\mathcal{H}(\zeta) = M^+(\zeta) \,/\, M^-(\zeta) = M^+(\zeta)\, M^+(-\zeta) \, , \qquad (6.16)$$

using (4.9). As may be shown by an expansion into an infinite product,[20]

$$Z_0(\gamma b) = \log \frac{b}{a} \prod_{n=1}^{\infty} \left(1 - \frac{\gamma^2}{\gamma_n^{\,2}} \right) \, . \qquad (6.17)$$

Substituting the explicit expressions of $\mathcal{H}(\zeta)$, using the expansion (6.17), and $M^+(\zeta)$ into (6.16), the common factor (6.17) may be cancelled from both sides. Now consider (6.16) for $|\zeta| \to \infty$ in the strip and substituting in the asymptotic forms of the Hankel functions, we have

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20 Ibid., pp. 136-7, Ref. (4).

$$\sqrt{\frac{a}{b}}\, e^{i(b-a)\sqrt{k^2-\zeta^2}} =$$

$$\frac{c^2}{\log \frac{b}{a}}\, e^{\frac{2i(b-a)}{\pi}\sqrt{k^2-\zeta^2}\left[\tan^{-1}\frac{\sqrt{k^2-\zeta^2}}{k+\zeta}+\tan^{-1}\frac{k+\zeta}{\sqrt{k^2-\zeta^2}}\right]},$$

where $F(\zeta)$ and $F(-\zeta) \to 1$. Thus

$$C = \left(\frac{a}{b}\right)^{\frac{1}{4}} \sqrt{\log \frac{b}{a}}\ . \tag{6.18}$$

Before writing $M^+(\zeta)$ in its final form it will be found convenient to reexpress $F(\zeta)$. Let $x = \gamma'$ in (6.6); then

$$F(\zeta) = \exp\left[\frac{1}{2\pi i}\int_0^\infty \log\left\{\left(\frac{\pi\frac{I_o(bx)}{K_o(bx)}-i}{\pi\frac{I_o(ax)}{K_o(ax)}-i}\right) e^{-2(b-a)x}\right\} \left(\frac{x}{(x^2+k^2-\zeta^2)} - \frac{\zeta\, x}{(x^2+k^2-\zeta^2)\sqrt{x^2+k^2}}\right) dx\right]. \tag{6.19}$$

Upon extending the range of integration, the first part of this integral becomes

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \log\left(\frac{K_o(ax)}{K_o(bx)}\, e^{-(b-a)x}\right) \frac{x dx}{(x^2+k^2-\zeta^2)} \tag{6.20}$$

$$= \tfrac{1}{2}\log\left(\sqrt{\frac{b}{a}}\, \frac{H_o^{(1)}(\gamma b)}{H_o^{(1)}(\gamma a)}\, e^{-i\gamma(b-a)}\right) .$$

where the integral is evaluated by closing the contour of integration in the lower half plane. The second part of the integral in (6.19) is reexpressed by replacing bx by x and ax by x. Finally, we obtain

$$F(\zeta) = \left(\sqrt{\frac{b}{a}}\,\frac{H_o^{(1)}(\gamma b)}{H_o^{(1)}(\gamma a)}\,e^{-i\gamma(b-a)}\right)^{\frac{1}{2}} \tag{6.21}$$

$$\times \exp\left[\frac{i\zeta}{2\pi}\int_0^\infty \log\left\{\left(\frac{\pi I_o(x)}{K_o(x)} - i\right) e^{-2x}\right\}\left(\frac{bx}{(x^2+b^2(k^2-\zeta^2))\sqrt{x^2+b^2k^2}}\right.\right.$$

$$\left.\left. - \frac{ax}{[x^2+a^2\,(k^2-\zeta^2)]\,\sqrt{x^2+a^2k^2}}\right) dx\right] .$$

This result could have been obtained directly by applying the general Couchy integral formula method to

$$\sqrt{\frac{b}{a}}\left(H_o^{(1)}(\gamma b)\,/\,H_o^{(1)}(\gamma a)\right)e^{-i\gamma(b-a)} .$$

As a check on this result we note that

$$F(\zeta)\,F(-\zeta) = \sqrt{\frac{b}{a}}\,\frac{H_o^{(1)}(\gamma b)}{H_o^{(1)}(\gamma a)}\,e^{-i\gamma(b-a)} .$$

As $|\zeta| \to \infty$, $|\mathrm{Im}\,\zeta| < \alpha$ this becomes 1 as it should; and as $|\zeta| \to k$ this becomes $\sqrt{\frac{b}{a}}$ which is correct.

Writing $M^+(\zeta)$ in polar form for $\operatorname{Im}\zeta = 0$, $|\zeta| \leq k$, we obtain from (6.21), (6.18), and (6.15)

$$|M^+(\zeta)| = \sqrt{|\mathcal{K}(\zeta)|}\,\exp\left[-\frac{(b-a)}{2}\zeta + \frac{\zeta}{2\pi}\int_0^\infty \tan^{-1}\left\{\frac{K_0(x)}{\pi I_0(x)}\right\}\left(\frac{bx}{[x^2+b^2(k^2-\zeta^2)]\sqrt{x^2+b^2k^2}} - \frac{ax}{[x^2+a^2(k^2-\zeta^2)]\sqrt{x^2+a^2k^2}}\right)dx\right] \tag{6.22}$$

and, letting $\delta_1(\zeta) = \arg\left\{M^+(\zeta)\right\}$,

$$\delta_1(\zeta) = \frac{(b-a)}{\pi}\left\{\zeta\left(\log\frac{2\pi}{k\beta(b-a)}+1\right) + 2\sqrt{k^2-\zeta^2}\left(\tan^{-1}\frac{\sqrt{k^2-\zeta^2}}{k+\zeta} - \frac{\pi}{4}\right)\right\}$$
$$+ \frac{1}{2}\left\{\tan^{-1}\left(-\frac{J_0(\gamma b)}{N_0(\gamma b)}\right) - \tan^{-1}\left(-\frac{J_0(\gamma a)}{N_0(\gamma a)}\right)\right\}$$
$$+ \sum_{n=1}^{\infty}\left(\frac{(b-a)\zeta}{n\pi} - \sin^{-1}\left\{\frac{\zeta}{\sqrt{\gamma_n^2-\gamma^2}}\right\}\right)$$
$$+ \frac{\zeta}{2\pi}\int_0^\infty \log\left\{\sqrt{1+\frac{\pi^2 I_0^2(x)}{K_0^2(x)}}\,e^{-2x}\right\}\left(\frac{bx}{[x^2+b^2(k^2-\zeta^2)]\sqrt{x^2+b^2k^2}} - \frac{ax}{[x^2+a^2(k^2-\zeta^2)]\sqrt{x^2+a^2k^2}}\right)dx\,. \tag{6.23}$$

To evaluate (6.22) and (6.23) at $|\zeta| = k$ examine the integral in (6.21) and let $|\zeta| \rightarrow k$, ($|\zeta| \leq k$ and ζ real). Breaking the interval of integration into two parts, the first from 0 to $(k^2 - \zeta^2)^{3/16}$ and the second from $(k^2 - \zeta^2)^{3/16}$ to ∞, the integrand of the first integral may be simplified by the approximation

$$\frac{bx}{\sqrt{x^2 + b^2k^2}} \approx \frac{x}{|k|}$$

and

$$\log \left\{ \left(\pi \frac{I_0(x)}{K_0(x)} - i \right) e^{-2x} \right\} \approx \log(-i) = -\frac{\pi i}{2} ;$$

and the integrand of the second may be simplified by the approximation

$$\frac{1}{x^2 + b^2(k^2 - \zeta^2)} \approx \frac{1}{x^2}$$

where the largest error is for $x = (k^2 - \zeta^2)^{3/16}$, or

$$\left| \frac{1}{(k^2 - \zeta^2)^{3/8}[1 + b^2(k^2 - \zeta^2)^{5/8}]} - \frac{1}{(k^2 - \zeta^2)^{3/8}} \right| \leq b(k^2 - \zeta^2)^{\frac{1}{4}} \rightarrow 0$$

for $|\zeta| \rightarrow k$. Thus we have

$$\lim_{|\zeta|\to k} \frac{i\zeta}{2\pi} \int_0^\infty \log \left\{\left(\frac{\pi I_o(x)}{K_o(x)} - i\right) e^{-2x}\right\} \left(\frac{bx}{[x^2 + b^2(k^2 - \zeta^2)]\sqrt{x^2 + k^2b^2}} - \frac{ax}{[x^2 + a^2(k^2 - \zeta^2)]\sqrt{x^2 + k^2a^2}} \right) dx$$

$$= \lim_{|\zeta|\to k} \left[\frac{\zeta}{|k|} \frac{i}{2\pi} \left(-\frac{\pi i}{2}\right) \int_0^{(k^2-\zeta^2)^{3/16}} \left[\frac{x}{x^2 + b^2(k^2 - \zeta^2)} - \frac{x}{x^2 + a^2(k^2 - \zeta^2)} \right] dx \right.$$

$$\left. + \frac{i\zeta}{2\pi} \int_{(k^2-\zeta^2)^{3/16}}^\infty \log \left\{\left(\frac{\pi I_o(x)}{K_o(x)} - i\right) e^{-2x}\right\} \left(\frac{b}{\sqrt{x^2 + b^2k^2}} - \frac{a}{\sqrt{x^2 + a^2k^2}} \right) \frac{dx}{x} \right]$$

$$= -\frac{k}{4|k|} \log \frac{b}{a} + \frac{i}{2\pi} \int_0^\infty \log \left\{\left(\frac{\pi I_o(x)}{K_o(x)} - i\right) e^{-2x}\right\} \left(\frac{bk}{\sqrt{x^2 + k^2b^2}} - \frac{ak}{\sqrt{x^2 + k^2a^2}} \right) \frac{dx}{x} \tag{6.23a}$$

where for $\zeta \to -k$ we replace $+k$ by $-k$ in the formula. Taking real and imaginary parts of (6.23a) and substituting the results into (6.22) and (6.23), we obtain

$$|M^+(k)| = \sqrt{\log \frac{b}{a}} \exp\left[-\frac{(b-a)k}{2} - \frac{k}{4|k|} \log \frac{b}{a} + \frac{k}{2\pi} \int_0^\infty \tan^{-1} \left\{\frac{K_o(x)}{\pi I_o(x)}\right\} \left(\frac{b}{\sqrt{x^2 + k^2b^2}} - \frac{a}{\sqrt{x^2 + k^2a^2}} \right) \frac{dx}{x} \right] \tag{6.24}$$

and

$$\delta_1(k) = \frac{(b-a)k}{\pi}\left(\log \frac{2\pi}{k\beta(b-a)} + 1\right) + \sum_{n=1}^{\infty}\left\{\frac{(b-a)k}{n\pi} - \sin^{-1}\frac{k}{\gamma_n}\right\}$$

$$+ \frac{k}{2\pi}\int_0^{\infty} \log\left\{\sqrt{1+\frac{\pi^2 I_o^2(x)}{K_o^2(x)}}\; e^{-2x}\right\}\left(\frac{b}{\sqrt{x^2+k^2b^2}} - \frac{a}{\sqrt{x^2+a^2k^2}}\right)\frac{dx}{x} .$$

When $k \to 0$ the integral in (6.23a) yields $+\frac{k}{4|k|}\log\frac{b}{a}$ by the same process so that

$$\lim_{k \to o} |M^+(k)| = \sqrt{\log \frac{b}{a}}$$

and $\lim_{k\to o} \delta_1(k) = 0$.

7. Verification that $M^+(\zeta)$ yields a solution.

It has already been shown in the previous work that $f(\zeta)$, defined by (4.3), is analytic in the finite ζ plane. Therefore $f(\zeta)$ is an integral function, i.e., is expressible as a power series in ζ. It remains to be shown that $f(\zeta)$ approaches a constant for $|\zeta| \to \infty$. First, for $M^+(\zeta)$ consider the asymptotic behavior of the infinite product given by

$$\prod_{n=1}\left(\sqrt{1-\frac{k^2}{\gamma_n^2}} - \frac{i\zeta}{\gamma_n}\right) e^{\frac{i\zeta(b-a)}{n\pi}} .$$

Substituting in the asymptotic form of γ_n and neglecting k/γ_n, this becomes

$$\prod_{n=1}^{\infty}\left(1-\frac{i\zeta(b-a)}{n\pi}\right)e^{\frac{i\zeta(b-a)}{n\pi}}$$

Using the Weirstrass definition[21] of the Γ function and Sterling's asymptotic formula[22] for Γ, the asymptotic form of the infinite product becomes

$$\frac{1}{\sqrt{-\frac{i\zeta(b-a)}{\pi}}}\exp\left[\frac{i\zeta(b-a)}{\pi}\left\{\log\left(-\frac{i\zeta(b-a)}{\pi}\right)+c-1\right\}\right]$$

where $c = \log\beta$ is Euler's constant. This is valid for the entire ζ plane except the negative imaginary axis. We also have the relations

$$\frac{2i(b-a)}{\pi}\sqrt{k^2-\zeta^2}\tan^{-1}\frac{\sqrt{k^2-\zeta^2}}{k+\zeta}=\frac{b-a}{\pi}\sqrt{k^2-\zeta^2}\log\left(\frac{\zeta+i\sqrt{k^2-\zeta^2}}{k}\right)$$

$$\sim -\frac{i\zeta}{\pi}(b-a)\log\frac{2\zeta}{k}\text{ for }-\frac{\pi}{2}\leq\arg\sqrt{k^2-\zeta^2}\leq\frac{\pi}{2}.$$

Then from (6.15) we get

$$M^{+}(\zeta)\sim\frac{1}{(-i\zeta)^{\frac{1}{2}}}\text{ for }|\zeta|\rightarrow\infty,\ \operatorname{Im}\zeta>-\alpha. \tag{7.1}$$

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21 Ibid., p. 236, Ref. (4)

22 Ibid., p. 253, Ref. (4).

Similarly we may find

$$M^{-}(\zeta) \sim (i\zeta)^{\frac{1}{2}} \text{ for } |\zeta| \rightarrow \infty \text{ , } \operatorname{Im}\zeta < \alpha. \qquad (7.2)$$

Second, the asymptotic behavior of $\mathcal{E}_z(\zeta)$ is to be found by examining $E_z(\rho, z)$ in the neighborhood of $\rho = b$ and $z = 0$. In this region we may regard the dimensions as very small compared with a wavelength, so that the static approximation may be used. Solving for the static field near the edge of a charged plate by using the Schwartz-Christoffel transformation, we have $E_z(b, z) \sim 1/z^{\frac{1}{2}}$ for $z \rightarrow 0+$ (where 0+ means z approaching zero from the positive side of the origin).[23] Using definition (3.1) the transform becomes

$$\mathcal{E}_z(\zeta) \sim 1/(i\zeta)^{\frac{1}{2}} \text{ for } |\zeta| \rightarrow \infty \text{ , } \operatorname{Im}\zeta < 0. \qquad (7.3)$$

Third to obtain the asymptotic behavior of $\Phi(\zeta)$, consider $\psi(b, z)$ defined by

$$\psi(b, z) = bH_\phi(\rho, z) \Big|_{\rho = b-0}^{\rho = b+0} . \qquad (7.4)$$

Since $H_\phi(\rho, z)$ is a continuous function of z for $\rho = b \pm 0$, there being no infinite current discontinuities, $\psi(b, z)$ is a continuous function of z and is zero for $z \geq 0$. Then $\psi(b, z) \sim z^{\beta}$ with $\beta > 0$ and $z \rightarrow 0^{-}$. Taking the Fourier transform of $\psi(b, z)$,

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[23] W. R. Symthe, "Static and Dynamic Electricity" (Mcgraw-Hill Co., New York, 1939), p. 82, Ref. (5).

$$\Psi(b, \zeta) \sim 1/(-i\zeta)^{\beta+1} \text{ for } |\zeta| \to \infty, \ \mathrm{Im}\,\zeta > 0 . \qquad (7.5)$$

The relation

$$\Psi(b, \zeta) = \frac{-ik\eta}{k^2 - \zeta^2}\, \bar{\Phi}(\zeta)$$

(where η appearing here is the so-called intrinsic admittance) may be deduced from equation (1.6).[24] Thus

$$\bar{\Phi}(\zeta) \sim (-i\zeta)^{1-\beta} \text{ for } |\zeta| \to \infty, \ \mathrm{Im}\,\zeta > 0. \qquad (7.6)$$

Combining (7.1), (7.2), (7.3), and (7.6), we obtain the result from (4.3)

$$f(\zeta) = \left\{ \begin{array}{ll} M^{+}(\zeta)\, \bar{\Phi}(\zeta) \sim (-i\zeta)^{\frac{1}{2}-\beta} & \text{for } \mathrm{Im}\,\zeta > 0 \\ \\ M^{-}(\zeta)\, \mathcal{E}_z(\zeta) \sim \text{const} & \text{for } \mathrm{Im}\,\zeta < 0 \end{array} \right\}, \qquad (7.7)$$

for $|\zeta| \to \infty$. Since β is greater than zero, $f(\zeta)$ can not become infinite in the upper half plane as rapidly as $(-i\zeta)^{\frac{1}{2}}$. Then $f(\zeta)$ is a polynomial of degree less than $\frac{1}{2}$, i.e., a constant, and β must equal $\frac{1}{2}$. Thus we have verified that equations (4.4) and (4.5) giving $\bar{\Phi}(\zeta)$ and $\mathcal{E}_z(\zeta)$ in turns of $M^{+}(\zeta)$ and $M^{-}(\zeta)$ are proper.

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24 Cf. post, p. 33, equation (8.1)

8. Integral expressions for $H_\phi(\rho, z)$

As will be seen in section 9, $H_\phi(\rho, z)$ is more useful for the evaluation of quantities of physical interest than is $E_z(\rho, z)$; so that it is now necessary to find $H_\phi(\rho, z)$ in terms of $E_z(\rho, z)$. Multiplying equation (1.5) by $e^{-i\zeta z}$ and integrating by parts with respect to z from $-\infty$ to $+\infty$, the left side becomes

$$\left[e^{-i\zeta z} \left(\frac{\partial H_\phi}{\partial z} - i\zeta H_\phi \right) \right]_{z=-\infty}^{z=+\infty} + (k^2 - \zeta^2) \int_{-\infty}^{\infty} e^{-i\zeta z} H_\phi dz .$$

Since asymptotically H_ϕ varies as $e^{ik|z|}$ where k is assumed to have a small imaginary part, α, the bracketted term is zero for $|\mathrm{Im}\,\zeta| < \alpha$. Defining the Fourier transform H_ϕ as

$$\mathcal{H}_\phi(\rho, \zeta) = \int_{-\infty}^{\infty} H_\phi(\rho, z)\, e^{-i\zeta z}\, dz ,$$

the relation between the transforms becomes

$$\mathcal{H}_\phi(\rho, \zeta) = \frac{ik\eta}{k^2 - \zeta^2} \frac{\partial \mathcal{E}_z(\rho, \zeta)}{\partial \rho} . \tag{8.1}$$

Substituting in the expression for $\mathcal{E}_z(\rho, \zeta)$ given by (4.6) and taking the inverse transform,

$$H_\phi(\rho, z) = -\frac{ck\eta}{2\pi i} \int_{-\infty}^{\infty} \frac{\frac{\partial \mathcal{K}(b, \rho, \zeta)}{\partial \rho}}{M^+(\zeta)(k^2 - \zeta^2)} e^{i\zeta z}\, d\zeta . \tag{8.2}$$

Consider the following regions:

1. $\rho \geqq b$, $z \geqq 0$,

2. $a \leqq \rho \leqq b$, $z \geqq 0$,

3. $\rho \geqq b$, $z \leqq 0$,

and 4. $a \leqq \rho \leqq b$, $z \leqq 0$, (see Fig. 7).

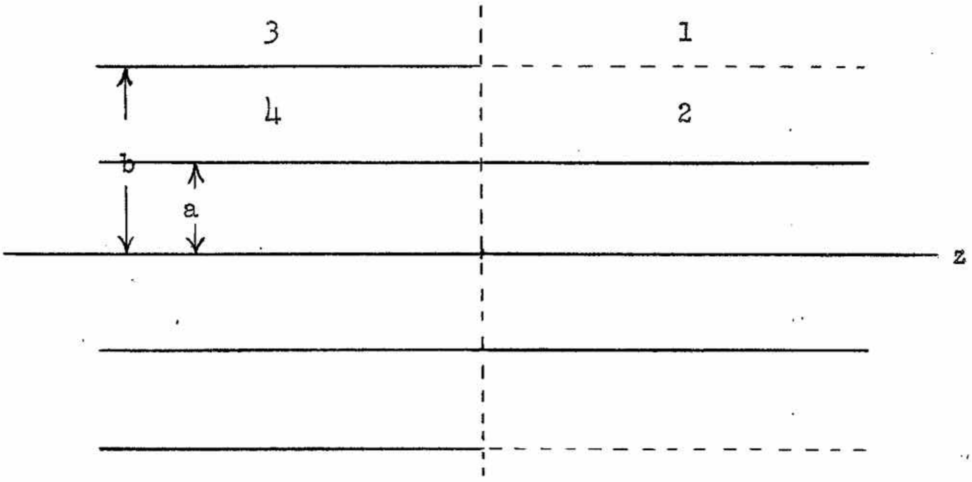

Fig. 7. Regions for which different expressions of H_ϕ are found.

The following expressions for $H_\phi(\rho, z)$ are obtained by choosing the proper form of $\mathcal{H}(b, \rho, \zeta)$, depending on ρ, and writing $M^+(\zeta)$ as $\mathcal{H}(b, b, \zeta)\, M^-(\zeta)$ for $z \leqq 0$:

for region 1

$$H_\phi = \frac{ck\eta}{2\pi i} \int_{-\infty}^{\infty} \frac{H_1^{(1)}(\gamma\rho) Z_0(\gamma b)}{H_0^{(1)}(\gamma a)\gamma M^+(\zeta)}\, e^{i\zeta z}\, d\zeta \; ; \qquad (8.3)$$

for region 2

$$H_\phi = -\frac{ck\eta}{2\pi i}\int_{-\infty}^{\infty}\frac{H_o^{(1)}(\gamma b)Z_o'(\gamma\rho)}{H_o^{(1)}(\gamma a)\gamma M^+(\zeta)}e^{i\zeta z}\,d\zeta \qquad (8.4)$$

where $Z_o'(\gamma\rho) = \frac{1}{\gamma}\frac{\partial}{\partial\rho}Z_o(\gamma\rho)$; (8.4a)

for region 3

$$H_\phi = \frac{ck\eta}{2\pi i}\int_{-\infty}^{\infty}\frac{H_1^{(1)}(\gamma\rho)e^{i\zeta z}}{H_o^{(1)}(\gamma b)\gamma M^-(\zeta)}\,d\zeta\ ; \qquad (8.5)$$

and for region 4

$$H_\phi = -\frac{ck\eta}{2\pi i}\int_{-\infty}^{\infty}\frac{Z_o'(\gamma\rho)e^{i\zeta z}}{Z_o(\gamma b)\gamma M^-(\zeta)}\,d\zeta\ . \qquad (8.6)$$

These expressions may be rewritten by closing the contour of integration, in the upper half plane for $z > 0$ and in the lower half plane for $z < 0$ (see Fig. 8). After some minipulation, we obtain:

for region 1

$$H_\phi = -\frac{2ck\eta}{\pi^2}\int_{k}^{\infty}\frac{Z_o(\gamma b)Z_o'(\gamma\rho)e^{i\zeta z}}{|H_o^{(1)}(\gamma a)|^2\gamma M^+(\zeta)}\,d\zeta\ ; \qquad (8.7)$$

for region 2

$$H_\phi = \frac{\eta c e^{ikz}}{2M^+(k)\rho} + \left\{\text{expression (8.7)}\right\} ; \qquad (8.8)$$

for region 3

$$H_\phi = \frac{ck\eta}{\pi} \int_{-k}^{-\infty} \frac{\left\{J_1(\gamma\rho)N_o(\gamma b) - N_1(\gamma\rho)J_o(\gamma b)\right\} e^{i\zeta z}}{\left|H_o^{(1)}(\gamma b)\right|^2 \gamma M^-(\zeta)} \, d\zeta \; ; \qquad (8.9)$$

and for region 4

$$H_\phi = \frac{c\eta \; e^{-ikz}}{2 \log \frac{b}{a} M^-(-k)\rho} + \left\{\text{attenuated terms}\right\} \qquad (8.10)$$

where

$$\left\{\text{attenuated terms}\right\} = \frac{2ck\eta}{\pi} \sum_{n=1}^{\infty} \left[\frac{Z_o'(h_n \rho) e^{\gamma_n z}}{h_n M^-(-iK_n)} \times \lim_{\zeta \to -iK_n} \frac{(\zeta + iK_n)}{Z_o(\gamma b)} \right] , \text{ where } h_n = \sqrt{k^2 + K_n^2}$$

are the roots of $Z_o(h_n b) = 0$.

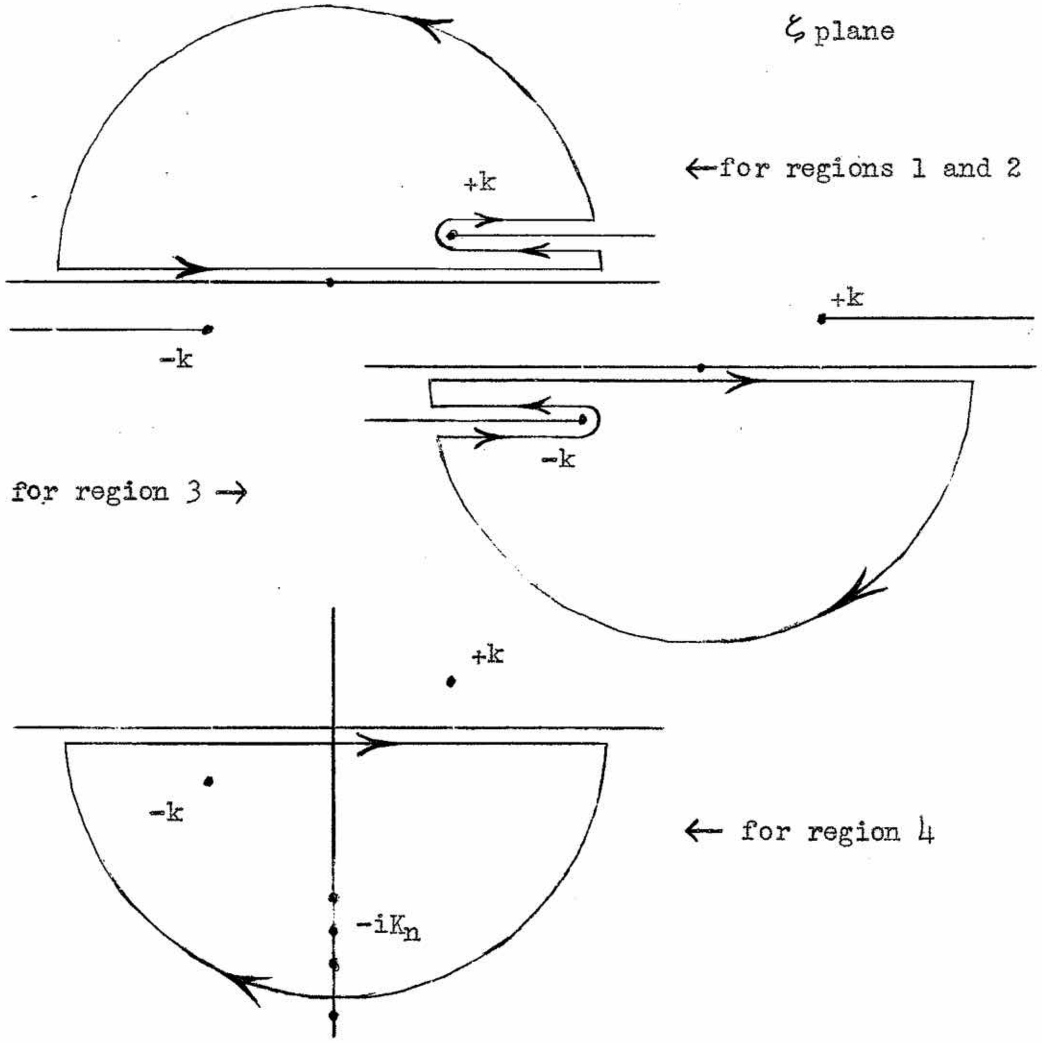

Fig. 8. Integration contours for H_ϕ.

We are now in a position to discuss the qualitative nature of our solution. It is immediately obvious that the solution lacks continuity between regions 1 and 2. Therefore our solution is not proper. However by adding a solution of the differential equation[25] satisfying the boundary condition (1.8),

[25] The differential equation in this case is

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2} + k^2\right) H_\phi = 0$$

(see Appendix A).

$$-\frac{c\eta e^{ikz}}{2M^{+}(k)\rho}, \tag{8.11}$$

to regions 2 and 4, continuity is established. The solution added is merely the original source that was assumed to exist inside the coaxial region for $z \rightarrow -\infty$. Levine and Schwinger presented a method [26] which may be used to simultaneously obtain the form of the solution for $z \rightarrow -\infty$ inside the coaxial region and the asymptotic form of the solution outside the coaxial region. Although the method is neat and elegant, it tends to obscure the fact that an incorrect solution has been obtained until the extra term (8.11) has been added. In conclusion, we may use H_ϕ as given by (8.3) or (8.7) for both regions 1 and 2; for region 3 we may use H_ϕ as given by (8.3), (8.5), or (8.9); and for region 4 H_ϕ becomes

$$-\frac{c\eta}{2\rho}\left[\frac{e^{ikz}}{M^{+}(k)} - \frac{e^{-ikz}}{\log\frac{b}{a}\,M^{-}(-k)}\right] + \text{attenuated terms.} \tag{8.12}$$

9. Physical quantities of interest.

It is not practical to describe the electromagnetic field in detail; and so only the features of primary interest will be considered. From the reflection coefficient, R_1, defined as B/A (from equation (1.2)), it is possible to deduce the total power

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26 Levine and Schwinger, op. cit., pp. 386-7, Ref. (2). By employing the function $\phi(\vec{r}) = J_1(k\rho\sin\theta')\,e^{-ikz\cos\theta'}$, for the present problem, the same procedure may be used to obtain similar results.

radiated and the parameters for the equivalent circuit of the open end. The gain function, $G_1(\theta)$, defined as the ratio of the time average power flow per unit solid angle as a function of the angle θ (for $r \to \infty$), over the total power radiated, describes the configuration of the radiating field outside the coaxial region.

a. Equivalent circuit[27].

Consider the coaxial region for $z \to -\infty$. From equation (1.2) and Ampere's law, $I = 2\pi \rho H_\phi$, we obtain the current flowing on the inside cylinder

$$I = 2\pi(Ae^{ikz} + Be^{-ikz}) . \tag{9.1}$$

From equation (1.1) and the definition of the potential between the inside and outside cylinders, $V = \int_a^b E_\rho \, d\rho$, we obtain

$$V = \frac{1}{\eta} \log \frac{b}{a} (Ae^{ikz} - Be^{-ikz}) . \tag{9.2}$$

It may be noted that equations (9.1) and (9.2) correspond in form to the expressions for the current and voltage in a transmission line. Thus it is possible to construct an equivalent circuit. Defining a phase factor ks_1 such that

$$R_1 = - |R_1| \, e^{2iks_1} , \tag{9.3}$$

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[27] Stratton, op. cit. pp. 549-50, Ref. (1).

we have

$$I = 2\pi A e^{ikz} \left\{ 1 - |R_1| \; e^{2ik(s_1 - z)} \right\}$$

and

$$V = \frac{1}{\eta} \log \frac{b}{a} \; Ae^{ikz} \left\{ 1 + |R_1| \; e^{2ik(s_1 - z)} \right\} .$$

Consider I and V evaluated at $z = s_1$. Using $z = s_1$ as the end of our equivalent line circuit, we find that it is terminated by a pure shunt conductance (see Fig. 9),

$$G \equiv \frac{I}{V} = \frac{2\pi\eta}{\log \frac{b}{a}} \left(\frac{1 - |R_1|}{1 + |R_1|} \right) . \qquad (9.4)$$

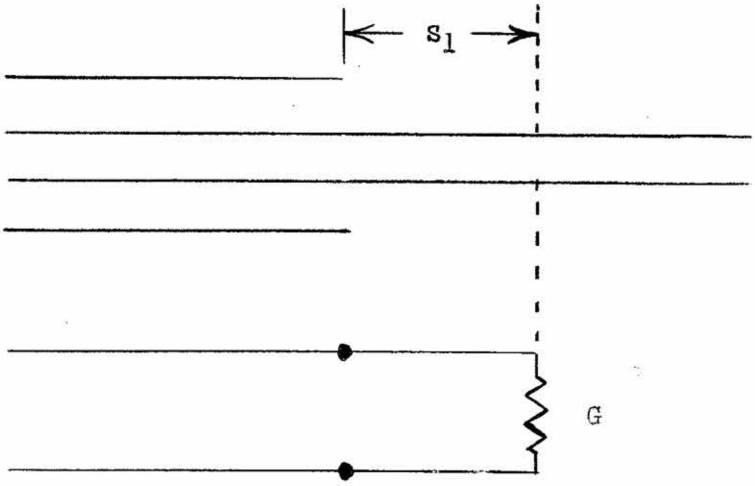

Fig. 9. The equivalent circuit.

The total time average power radiated is given by

$$P_{t1} = \frac{1}{2} \operatorname{Re} \left\{ I \widetilde{V} \right\} = \frac{\pi |A|^2}{\eta} \log \frac{b}{a} \left(1 - |R^2| \right) \quad . \qquad (9.5)$$

b. Expressions for $|R_1|$, s_1, and P_t.

From (8.12) we find that

$$R_1 = - \frac{M^+(k)}{\log \frac{b}{a} M^-(-k)} . \tag{9.6}$$

But from (4.9), we have $M^-(-k) = 1/M^+(k)$, so that

$$R_1 = - \frac{(M^+(k))^2}{\log \frac{b}{a}} . \tag{9.7}$$

Taking the absolute value

$$|R_1| = \frac{|M^+(k)|^2}{\log \frac{b}{a}} ; \tag{9.8}$$

and

$$ks_1 = \arg \ M^+(k) \ = \delta_1(k) . \tag{9.9}$$

The total time average power radiated, P_t, as given by (9.5) may be rewritten as follows, using (8.12) for A and (9.8) for $|R_1|$:

$$P_{t1} = \frac{\pi c^2 \eta}{4|R_1|} (1 - |R_1|^2) . \tag{9.10}$$

c. Gain function, $\mathcal{G}_1(\theta)$.

The gain function may be written as $P_1(\theta)/P_{t1}$ where

$$P_1(\theta) = \lim_{r\to\infty} r^2 \vec{S}_{av} \cdot \vec{n} \,,$$

where $\vec{S}_{av}$ is the time average Poynting's vector and $\vec{n}$ is the unit vector normal to the sphere of radius r. Using the definition, $\vec{S}_{av} = \frac{1}{2} \operatorname{Re} \left\{ \vec{E} X \widetilde{\vec{H}} \right\}$ and Maxwell's second equation, $\vec{E} = - \nabla X \vec{H}/ik\eta$, we find

$$\mathcal{G}_1(\theta) = \frac{1}{2ik\eta P_{t1}} \lim_{r\to\infty} \left(r \operatorname{Re} \left\{ \frac{\partial}{\partial r} (rH_\phi) \widetilde{H}_\phi \right\} \right) \,. \qquad (9.11)$$

In the limit as $r \to \infty$, assume H_ϕ has the asymptotic form

$$H_\phi = f(\theta) \frac{e^{ikr}}{r} \,. \qquad (9.12)$$

Then substituting in the value of P_{t1} given by (9.10),

$$\mathcal{G}_1(\theta) = \frac{2|R_1|}{\pi \eta^2 c^2 (1 - |R_1|^2)} \lim_{r\to\infty} r^2 |H_\phi|^2 \,. \qquad (9.13)$$

Next we consider expression (8.3) which is valid anywhere outside the coaxial region for $\rho \to \infty$ and $|z| \to \infty$. Introducing the asymptotic form of the Hankel function and changing coordinates to r and θ, we have

$$H_\phi \approx -\frac{ck\eta}{2\pi}\sqrt{\frac{2}{\pi r \sin\theta}}\, e^{-\frac{i\pi}{4}} \int_{-\infty}^{\infty} \frac{Z_o(\gamma b)e^{ir(\sqrt{k^2-\zeta^2}\sin\theta+\zeta\cos\theta)}}{H_o^{(1)}(\gamma a)M^+(\zeta)(k^2-\zeta^2)^{3/4}}\, d\zeta .$$

For $r \to \infty$, this integral may be evaluated by the method of stationary phase.[28] Expanding the exponent about the zero of its first derivative, which is at $\zeta = k\cos\theta$, the exponent becomes

$$ikr\left[1 - \frac{(\zeta - k\cos\theta)^2}{2k^2\sin^2\theta} + 0\left\{(\zeta - k\cos\theta)^3\right\}\right] .$$

The slowly varying part of the integrand is evaluated at $\zeta = k\cos\theta$. For the remaining exponential term, let

$$x = (\zeta - k\cos\theta)\sqrt{\frac{kr}{2k^2\sin^2\theta}} .$$

Noting that $\int_{-\infty}^{\infty} e^{-ix^2}dx = \sqrt{\pi}\, e^{-\frac{i\pi}{4}}$, we get

$$H_\phi \approx \frac{ic\eta\, Z_o(k\, b\sin\theta)}{\pi H_o^{(1)}(k\, a\sin\theta)M^+(k\cos\theta)\sin\theta}\,\frac{e^{ikr}}{r} . \qquad (9.14)$$

Substituting this result into (9.13), we get

$$\mathcal{G}_1(\theta) = \frac{2|R_1|Z_o^2(k\, b\sin\theta)}{\pi^3(1-|R_1|^2)|H_o^{(1)}(k\, a\sin\theta)|^2|M^+(k\cos\theta)|^2\sin^2\theta} . \quad (9.17)$$

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28 G. N. Watson, "Treatise on the Theory of Bessel Functions," (Cambridge Press, Macmillan Co., New York, 1948), pp. 229-30; Ref. (6).

To check that $\mathcal{G}_1(\theta)$ is properly normalized, we require that

$$2\pi \int_0^{\pi} \mathcal{G}_1(\theta) \sin \theta \, d\theta = 1 . \qquad (9.18)$$

Although $\mathcal{G}_1(\theta)$ becomes infinite for $\theta = 0$ or π, the integral converges. See Appendix C for the proof that (9.18) holds.

Chapter II

The case of a single half infinite cylinder

The problem of a single half infinite cylinder may be stated and solved exactly, although the problem doesn't properly fit any actual physical situation as may be seen from Section 17. This problem is treated independently of the problem of a finite inside cylinder, but the results of this chapter together with the results of Chapter I are used in Chapter III to obtain approximate solutions for the case of a finite inside cylinder. No numerical results are obtained for the problem of a single half infinite cylinder, inasmuch as the desired physical parameters are not properly defined.

1. Statement of the problem.

The metallic boundaries of the region being considered consist of a half infinite circular cylinder radius $\underline{a}$ and of vanishing thickness, extending from $z \to -\infty$ to $z = 0$. Coordinates are chosen as indicated in Fig. 10.

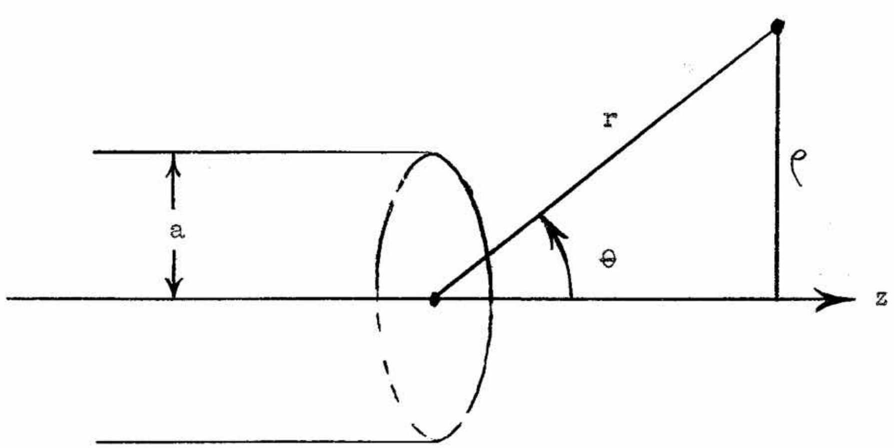

Fig. 10. The half infinite circular cylinder.

A source is assumed to exist outside the cylinder for $z \to -\infty$ of the form

$$E_\rho = \frac{1}{\eta}\,\frac{A\,e^{ikz}}{\rho}\ , \tag{10.1}$$

$$H_\phi = \frac{Ae^{ikz}}{\rho}\ , \tag{10.2}$$

and

$$E_z,\ E_\phi,\ H_z,\ H_\rho = 0 \tag{10.3}$$

For a vacuum (or air), ideal metallic boundaries, time harmonic field components, and axial symmetry, Maxwell's equations reduce to (1.4), (1.5) and (1.6). The boundary conditions in this case are

$$E_z = 0 \quad \text{for } \rho = a,\ z \leq 0 \tag{10.4}$$

and

$$\left(\frac{1}{\rho} + \frac{\partial}{\partial\rho}\right) H_\phi = 0 \quad \text{for } \rho = a,\ z \leq 0\ . \tag{10.5}$$

The radiation condition (1.9) holds also for the present case. We specify that

$$k < \frac{u_{nm}}{a}\ ,\ (u_{nm} = 1.84\ ,\ 2.40\ ,\ \ldots) \tag{10.6}$$

where u_{nm} are the roots of $J_n(u) = 0$ or $J_n'(u) = 0$; in order that no propagation may occur inside the cylinder.[1]

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[1] Stratton, op. cit., pp. 537-9, Ref. (1).

11. Derivation of the integral equation.

Consider the free space Green's function,

$$G(\vec{r},\vec{r}') = \frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}, \qquad (11.1)$$

which satisfies equation (2.1). Applying Green's second identity to $E_z(\vec{r})$ and $G(\vec{r},\vec{r}')$, we have

$$\int_{\text{Volume}} \left\{G(\vec{r}',\vec{r})(\nabla'^2+k^2)E_z(\vec{r}')-E_z(\vec{r}')(\nabla'^2+k^2)G(\vec{r}',\vec{r})\right\}d\tau'$$

$$= \int_{\text{Surfaces}} \left\{G(\vec{r}',\vec{r})\,\frac{\partial E_z(\vec{r}')}{\partial n'} - E_z(\vec{r}')\,\frac{\partial G(\vec{r}',\vec{r})}{\partial n'}\right\}da'. \qquad (11.2)$$

From equations (1.4) and (2.1) the volume integral on the left becomes merely $E_z(\vec{r})$. For the surface integral on the right, we break up the surfaces as follows: S_1 is the surface inside the cylinder for $z \to -\infty$; S_2 is the inside surface of the cylinder; S_3 is the outside surface of the cylinder; and S_4 is the remainder of the surface of the sphere at infinity (see Fig. 11).

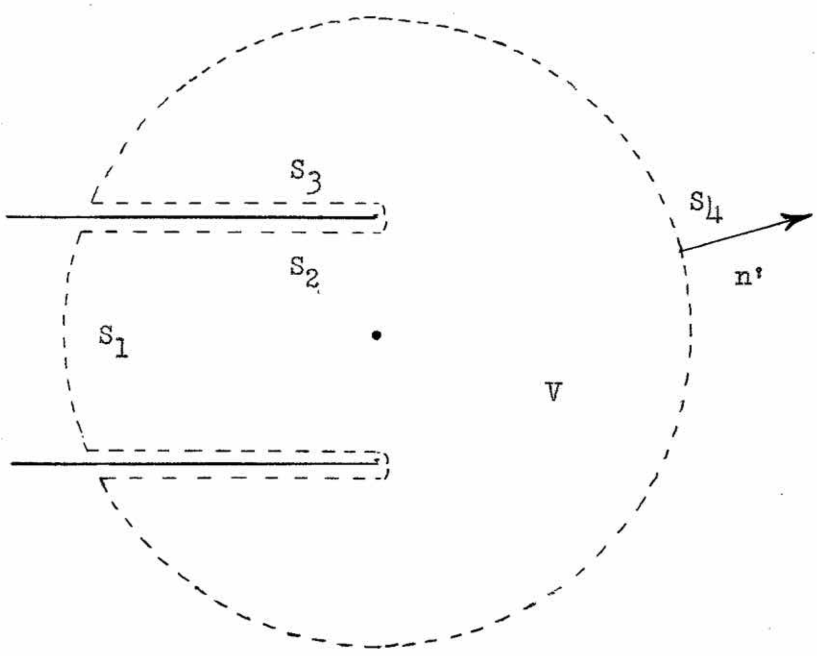

Fig. 11. The region for Green's second identity.

The behavior of the integrand on these surfaces may be specified as follows: on S_1, $E_z \equiv 0$ from (10.6); on S_2 and S_3, $E_z = 0$ from (10.4); and on S_4 the integrand vanishes due to conditions (1.9) and (11.1) and the assumption that $k = \beta + i\alpha$ where α is an arbitrarily small positive quantity. Equation (11.2) then reduces to

$$E_z(\rho, z) = \int_{-\infty}^{0} G(a, \rho, z - z') \, \phi(a, z') dz' \qquad (11.3)$$

where

$$\phi(a, z') = - a \frac{\partial E_z}{\partial \rho'} (\rho', z') \Big|_{\rho' = a-0}^{\rho' = a+0} . \qquad (11.4)$$

Applying the boundary conditions (10.4), we obtain the integral equation

$$E_z(a,z) = \int_{-\infty}^{\infty} G(a,a,z-z')\phi(a,z')dz' \qquad (11.5)$$

where

$$E_z(a,z) = \begin{cases} 0 & \text{for } z \leq 0 \\ E_z(a,z) & \text{for } z > 0 \end{cases}$$

and

$$\phi(a,z) = \begin{cases} \phi(a,z) & \text{for } z \quad 0 \\ 0 & \text{for } z = 0 \end{cases}.$$

This resembles the Wiener-Hopf type of integral equation, and may be solved by a Fourier transform method. The object is to solve (11.5) for the unknown function, $\phi(a,z')$; and, having found $\phi(a,z')$, to substitute it into equation (11.3) to obtain $E_z(\rho,z)$.

12. Derivation of the transform equation.

The transforms of $E_z(\rho,z)$ and $\phi(a,z)$ are defined as follows:

$$\mathcal{E}_z(\rho,\zeta) = \int_{-\infty}^{\infty} E_z(\rho,z)e^{i\zeta z}\, dz \qquad (12.1)$$

and

$$\phi(a, \zeta) = \int_{-\infty}^{\infty} \phi(a,z)\, e^{-i\zeta z}\, dz = \int_{-\infty}^{0} \phi(a,z)\, e^{-i\zeta z}\, dz . \quad (12.2)$$

Multiplying equation (11.3) by $e^{-i\zeta z}$, integrating from $z = -\infty$ to $z = +\infty$, and assuming that it is possible to change the order of integration, we obtain

$$\int_{-\infty}^{\infty} E_z(\rho, z)\, e^{-i\zeta z}\, dz$$

$$= \int_{-\infty}^{\infty} G(a, \rho, z-z')e^{-i\zeta(z-z')}dz \int_{-\infty}^{0} \phi(a, z')e^{-iz'\zeta}dz' . \quad (12.3)$$

Substituting the definitions of the transforms into (12.3), we get

$$\mathcal{E}_z(\rho, \zeta) = \mathcal{G}(a, \rho, \zeta)\, \phi(a, \zeta) . \quad (12.4)$$

According to the convolution theorem, this process is only valid for the common domain of analyticity of the transforms. In particular, we are interested in the transform equation for $\rho = a$. Applying boundary condition (10.4) to the definition, (12.1), we obtain

$$\mathcal{E}_z(a, \zeta) = \int_{0}^{\infty} E_z(a, z)\, e^{-i\zeta z}\, dz . \quad (12.5)$$

And applying the boundary condition, (10.4), to (12.4), we get

$$\mathcal{E}_z(a, \zeta) = \mathcal{G}(a,a,\zeta)\ \Phi(a,\zeta)\ ;$$

or merely

$$\mathcal{E}_z(\zeta) = \mathcal{G}(\zeta)\ \Phi(\zeta)\ . \qquad (12.6)$$

Equation (12.6) is to be solved for the unknown function $\Phi(\zeta)$. The solution, $\Phi(\zeta)$, is then substituted into (12.4) to give $\mathcal{E}_z(\rho, \zeta)$ from which $E_z(\rho, z)$ may be obtained by taking the inverse transform.

In order to find the region in which equation (12.6) is valid, we determine the regions in which the transforms $\mathcal{G}(\zeta)$, $\Phi(\zeta)$, and $\mathcal{E}_z(\zeta)$ are defined. From Appendix D, we have

$$\mathcal{G}(\rho_>, \rho_<, \zeta) = \frac{\pi i}{2} H_0^{(1)}(\gamma \rho_>)\ J_0(\gamma \rho_<) \qquad (12.7)$$

which is defined in the strip $|\mathrm{Im}\,\zeta| < \alpha$; therefore $\mathcal{G}(\zeta)$ is analytic in the strip $|\mathrm{Im}\,\zeta| < \alpha$. For $\Phi(\zeta)$, we substitute the asymptotic form of $\phi(a,z)$ for $z \rightarrow -\infty$ into the integrand of (12.2). Since E_z is taken to be identically zero for the initial waves traveling from the left, only waves scattered from the end of the cylinder contribute to the E_z component for $z \rightarrow -\infty$. The integrand of (12.2) varies as

$$e^{-ikz-i\zeta z} \quad \text{for } z \rightarrow -\infty$$

Then

$$|\Phi(\zeta)| \leq \int_{-\infty}^{0} |\phi(a,z)| \; e^{\eta z} \, dz$$

is bounded for Im $\zeta > -\alpha$, since $|\phi(a,z)|$ remains finite. Therefore $\Phi(\zeta)$ is analytic in the upper half plane, Im $\zeta > -\alpha$. For $\mathcal{E}_z(\zeta)$, we consider the asymptotic behavior of $E_z(a,z)$ for $z \rightarrow +\infty$. The integrand of (12.5) varies as

$$e^{ikz-i\zeta z} \quad \text{for } z \rightarrow +\infty$$

Then

$$|\mathcal{E}_z(\zeta)| \leq \int_{0}^{\infty} |E_z(a,z)| \; e^{\eta z} \, dz$$

is bounded for Im $\zeta < \alpha$, since $|E_z(a,z)|$ remains finite except for $z = 0$ where it has an integrable singularity. Therefore $\mathcal{E}_z(\zeta)$ is analytic in the lower half plane Im $\zeta < \alpha$. These results are summarized in Fig. 12.

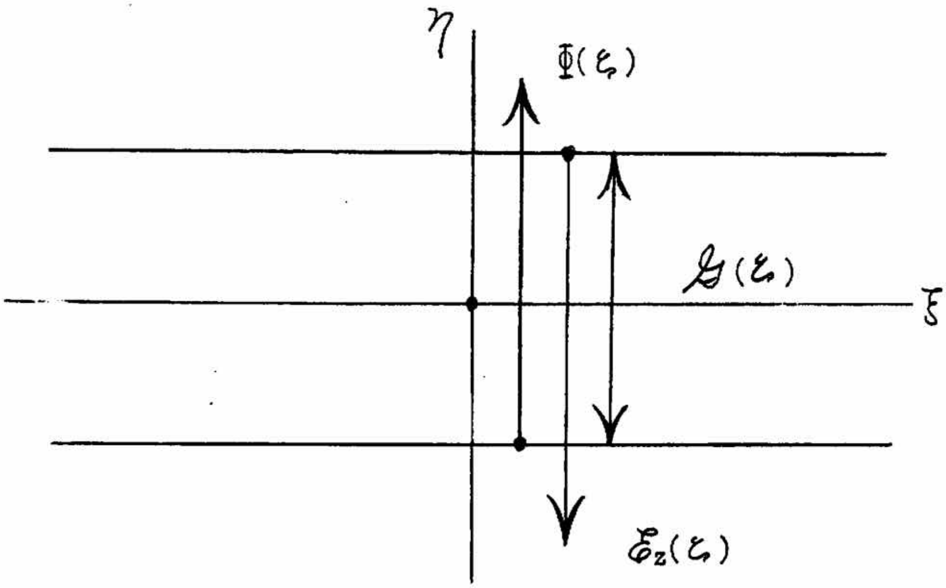

Fig. 12. Regions of analyticity of the transforms.

It has been shown that a common region of analyticity exists for which equation (12.6) is valid, i.e., the strip, $|\text{Im}\,\zeta| < \alpha$. It may also be shown that the transform equation (12.4) is valid in the strip $|\text{Im}\,\zeta| < \alpha$.

13. Method of solution of the transform equation.

The procedure for solving the transform equation, (12.6), is identical to the previous case of an infinite inside cylinder given in Section 4. We write $\mathcal{G}(\zeta)$ as a ratio of two functions $L^+(\zeta)/L^-(\zeta)$ such that $L^+(\zeta)$ is analytic and has no zeros in the upper half plane, $\text{Im}\,\zeta > -\alpha$, and $L^-(\zeta)$ is analytic and has no zeros in the lower half plane, $\text{Im}\,\zeta < \alpha$. Then

$$g(\zeta) \equiv \begin{cases} L^+(\zeta)\ \phi(a, \zeta) & \text{for } \operatorname{Im}\zeta > -\alpha \\ \text{Either expression for } |\operatorname{Im}\zeta| < \alpha \\ L^-(\zeta)\ \mathcal{E}_z(a, \zeta) & \text{for } \operatorname{Im}\zeta < \alpha \end{cases} \tag{13.1}$$

defines a rational integral function. Moreover, if $\lim_{\zeta\to\infty} g(\zeta)$ exists then $g(\zeta) \equiv C$; and[3]

$$\Phi(\zeta) = C/L^+(\zeta) . \tag{13.2}$$

To find $L^+(\zeta)$, we apply Cauchy's integral formula,

$$\log \mathcal{G}(\zeta) = \frac{1}{2\pi i}\oint \frac{\log \mathcal{G}(t)}{t - \zeta}\, dt , \tag{13.3}$$

for the rectangular path given in Section 4, Fig. 5. To show that the contribution over the ends vanishes, consider the asymptotic form[4] of $\mathcal{G}(t)$,

$$-\frac{e^{2i\gamma a - \frac{\pi i}{2}} + 1}{2i\gamma a} \text{ for } \gamma \to \infty \tag{13.4}$$

where $\gamma = \sqrt{k^2 - t^2}$. On the right $\gamma \sim it$ and on the left $\gamma \sim -it$ for $t \to \infty$; so that the integrand of (13.3) varies as

$$\frac{\log\left(\frac{e^{\mp 2ta - \frac{\pi i}{2}} + 1}{\pm 2ta}\right)}{ta} \sim \pm 2 \quad \text{for } |t| \to \infty ,$$

---

[3] Cf. post, Section 4, pp. 12-13.

[4] See Appendix D.

$|\text{Im } t| < \alpha$. When $\epsilon \to 0$, the contribution over the ends of the rectangle vanish. The expression for $L^+(\zeta)$ then becomes

$$L^+(\zeta) = 1/L^-(-\zeta) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log \mathcal{G}(t)}{t-\zeta}\, dt\right] \qquad (13.5)$$

provided, of course, the integral converges. There is an important difference between the present case and the previous one of the infinite inside cylinder. For the previous Green's function transform, $\mathcal{K}(\pm k)$ was finite. Here[5]

$$\mathcal{G}(\zeta) \approx \log \frac{2}{a\beta\sqrt{k^2-\zeta^2}} \quad \text{for } |\zeta| \to k ,$$

and we may deduce the following relations:

$$\lim_{\zeta \to +k} L^+(\zeta) \text{ exists and equals } L^+(k) ;$$

$$\lim_{\zeta \to -k} L^-(\zeta) \text{ exists and equals } L^-(-k) ;$$

$$L^+(k) = 1/L^-(-k) ; \qquad (13.6)$$

$$L^+(\zeta) \sim L^-(-k) \log \frac{2}{a\beta\sqrt{2k(k+\zeta)}} \text{ for } \zeta \to -k ; \qquad (13.7)$$

and

$$L^-(\zeta) \sim L^+(k)/\log \frac{2}{a\beta\sqrt{2k(k-\zeta)}} \text{ for } \zeta \to +k , \qquad (13.8)$$

or $L^-(k) = 0$.

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[5] Loc. cit.

14. Evaluation of $L^+(\zeta)$ by the second method

The first method used in the previous problem, Section 5, does not yield adequate results for the present problem. The procedure for the present case is identical to the procedure in Section 6. Considering

$$\frac{d}{d\zeta} \log L\,(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log \mathcal{G}(t)}{(t-\zeta)^2}\, dt \tag{14.1}$$

and breaking up the path of integration as shown in Fig. 6, the integral over Γ_1 reduces to

$$-\frac{1}{2\pi i} \int_{k}^{t} \frac{\log\left(1 + i\pi \frac{I_0(a\gamma')}{K_0(a\gamma')}\right)}{(t+\zeta)^2}\, dt \tag{14.2}$$

where $\gamma' = \sqrt{t^2 - k^2}$. Since the integrand varies as

$$\frac{\log e^{2a\gamma' - \frac{\pi i}{2}}}{t^2} \sim \frac{1}{t} \quad \text{for } t \to \infty ,$$

the integral diverges logarithmically for $T \to \infty$. In order to isolate this singularity add a quantity which is zero to the integrand,

$$\frac{\log e^{-2a\gamma' - \frac{\pi i}{2}}}{(t+\zeta)^2} + \frac{2a\sqrt{t^2 - k^2} + \frac{\pi i}{2}}{(t+\zeta)^2} .$$

From the result given by (6.5) the integral over Γ_1 becomes

$$
-\frac{1}{2\pi i}\int_{k}^{\infty}\log\left\{\left(\frac{\pi I_o(a\gamma')}{K_o(a\gamma')}-i\right)e^{-2a\gamma'}\right\}\frac{dt}{(t+\zeta)^2}\;\frac{1}{4(k+\zeta)}
$$

$$
-\frac{ia}{\pi}\left\{1+\frac{2\zeta}{\sqrt{k^2-\zeta^2}}\tan^{-1}\frac{\sqrt{k^2-\zeta^2}}{k+\zeta}\right\}+\frac{ia}{\pi}\lim_{T\to\infty}\log\frac{2T}{k}\,. \tag{14.3}
$$

In order to evaluate the integral over Γ_2 close the path with circular arcs at infinity and lines passing on both sides of a branch cut which is taken along the negative imaginary axis through the zeros of $J_o(a\gamma_n) = 0$ at $t_n = -i\sqrt{\gamma_n^2 - k^2}$. Since ζ is in the upper half plane, this contour incloses no singularities; and we have relation (6.8). For the value of the integral over the arcs consider the asymptotic form of the integrand. In the lower half plane for $|t| \to \infty$, $\sqrt{k^2 - t^2} \to it$; so that from (13.4) we have

$$
\mathcal{G}(t) \sim \frac{e^{-2at-\frac{\pi i}{2}}+1}{2at} \quad \text{for } |t| \to \infty,\ \operatorname{Im} t < \alpha.
$$

Since only the real part of t need be considered, it can be seen that there will only be a contribution over the arc in the third quadrant. In the limit as $T \to \infty$

$$
\frac{1}{2\pi i}\int_{\text{arcs}}\frac{\log \mathcal{G}(t)}{(t-\zeta)^2}\,dt = \frac{1}{2\pi i}\int_{-\frac{\pi}{2}}^{(-\pi)}\left(-\frac{2a}{t}\right)dt = \frac{a}{2} \tag{14.4}
$$

The integral over the branch cut is evaluated in an identical manner to the previous case shown in Section 6. Here the asymptotic form of the roots of $J_0(a\gamma_n) = 0$ are given by

$$\gamma_n = \frac{\left(n\pi - \frac{\pi}{4}\right)}{a} + 0\left(\frac{1}{n}\right)$$

The integral over the branch cut is found to yield

$$-\frac{ia}{\pi} - \sum_{n=1}^{\infty} \left(\frac{ia}{n\pi} + \frac{1}{\zeta + i\sqrt{\gamma_n^{\,2} - k^2}}\right) + \frac{ia}{\pi} \lim_{T\to\infty} \log \frac{a}{\pi} T \beta \quad (14.5)$$

where $\log \beta = 0.5772 \cdots$, is Euler's constant. Collecting results (14.3), (14.4), and (14.5) and combining them according to (6.2′) and (6.8), we obtain

$$\frac{d}{d\zeta} \log L^+(\zeta) = \frac{d}{d\zeta} \log P(\zeta) - \frac{1}{4(k+\zeta)} \quad (14.6)$$

$$+ \frac{ia}{\pi}\left\{\log \frac{2\pi i}{ka\beta} - \frac{2\zeta}{\gamma} \tan^{-1} \frac{\gamma}{k+\zeta}\right\}$$

$$+ \sum_{n=1}^{\infty} \left(\frac{ia}{n\pi} + \frac{1}{\zeta + i\sqrt{\gamma_n^{\,2} - k^2}}\right) .$$

where

$$P(\zeta) = \exp\left[\frac{1}{2\pi i}\int_k^{\infty} \log\left\{\left(\pi\frac{I_0(a\gamma')}{K_0(a\gamma')} - i\right) e^{-2a\gamma'}\right\} \frac{dt}{t+\zeta}\right] \quad (14.7)$$

Integrating (14.6) and taking the antilogarithm, we get

$$L^{+}(\zeta) = \frac{CP(\zeta)}{(k+\zeta)^{\frac{1}{4}}} \prod_{n=1}^{\infty}\left(\sqrt{1-\frac{k^2}{\gamma_n^{\,2}}} - \frac{i\zeta}{\gamma_n}\right) e^{\frac{ia\zeta}{n\pi}} \tag{14.8}$$

$$\times \exp\left[\frac{ia}{\pi}\,\zeta\left(\log\frac{2\pi i}{ka\beta}+1\right) + \frac{2ia\gamma}{\pi}\tan^{-1}\frac{\gamma}{k+\zeta}\right]$$

where C is a constant of integration. In order to determine C, it may be noted that in the strip $|\mathrm{Im}\,\zeta| < \alpha$

$$\mathcal{G}(\zeta) = L^{+}(\zeta)\, L^{+}(-\zeta) \tag{14.9}$$

from (13.5). Substituting the explicit expressions into (14.9), we get

$$\frac{\pi i}{2}\, J_o(\gamma a)\, H_o^{(1)}(\gamma a) = \frac{C^2 P(\zeta)P(-\zeta)}{(k^2-\zeta^2)^{\frac{1}{4}}} \prod_{n=1}^{\infty}\left(1-\frac{k^2-\zeta^2}{\gamma_n^{\,2}}\right) e^{ia\gamma}\,. \tag{14.10}$$

But from the expansion of $J_o(\gamma a)$ as an infinite product,[6]

$$J_o(\gamma a) = \prod_{n=1}^{\infty}\left(1-\frac{\gamma^2}{\gamma_n^{\,2}}\right)$$

Equation (14.10) then reduces to

$$\frac{\pi i}{2}\, H_o^{(1)}(\gamma a) = C^2\, \frac{P(\zeta)P(-\zeta)}{(k^2-\zeta^2)^{\frac{1}{4}}}\, e^{ia\gamma}$$

Considering this equation for $|\zeta| \to \infty$, $|\mathrm{Im}\,\zeta| < \alpha$,

---

[6] Watson, op. cit., p. 498, Ref. (6)

$$C = \left(\frac{\pi i}{2a}\right)^{\frac{1}{4}} \tag{14.11}$$

since $P(\zeta)$ and $P(-\zeta) \to 1$ for Re $\zeta \to \infty$. We may rewrite $P(\zeta)$ by changing the variable of integration, $x = a\gamma'$; so that

$$P(\zeta) = \exp\left[\frac{1}{2\pi i}\int_0^\infty \log\left\{\left(\frac{\pi I_0(x)}{K_0(x)} - i\right) e^{-2x}\right\}\right.$$

$$\left.\times\left(1 - \frac{\zeta a}{\sqrt{x^2 + k^2a^2}}\right)\frac{xdx}{[x^2 + a^2(k^2 - \zeta^2)]}\right]. \tag{14.12}$$

If the first integral in (14.12) is evaluated by extending the range of integration and closing in the lower half plane,

$$P(\zeta) = \left(\sqrt{\frac{\pi a\gamma}{2}}\, H_0^{(1)}(a\gamma)e^{-ia\gamma + \frac{\pi i}{4}}\right)^{\frac{1}{2}} \tag{14.13}$$

$$\times \exp\left[-\frac{a\zeta}{2\pi i}\int_0^\infty \log\left\{\left(\pi\frac{I_0(x)}{K_0(x)} - i\right) e^{-2x}\right\}\right.$$

$$\left.\times\frac{xdx}{\sqrt{x^2 + k^2b^2}\,[x^2 + a^2(k^2 - \zeta^2)]}\right].$$

Writing $L^+(\zeta)$ in polar form, substituting (14.12), (14.11) into (14.8), we get

$$\left|L^+(\zeta)\right| = \left(\frac{\pi J_0^2(\gamma a)}{2a(k+\zeta)}\right)^{\frac{1}{4}} \exp\left[-\frac{a\zeta}{2}\right. \qquad (14.14)$$

$$\left. - \frac{1}{2\pi}\int_0^\infty \tan^{-1}\left\{\frac{K_0(x)}{\pi I_0(x)}\right\}\left(1 - \frac{\zeta a}{\sqrt{x^2+k^2a^2}}\right)\frac{x\,dx}{[x^2+a^2(k^2-\zeta^2)]}\right] .$$

And, letting $\delta_2(\zeta) = \arg\left\{L^+(\zeta)\right\}$,

$$\delta_2(\zeta) = \frac{\pi}{8} + \frac{a}{\pi}\,\zeta\left(\log\frac{2\pi}{ka\beta} + 1\right) + \frac{2a\gamma}{\pi}\tan^{-1}\frac{\gamma}{k+\zeta}$$

$$+ \sum_{n=1}^{\infty}\left(\frac{a\zeta}{n\pi} - \sin^{-1}\frac{\zeta}{\sqrt{\gamma_n^2-\gamma^2}}\right) \qquad (14.15)$$

$$- \frac{1}{2\pi}\int_0^\infty \log\left\{\sqrt{1+\left(\frac{\pi I_0(x)}{K_0(x)}\right)^2}\, e^{-2x}\right\}\left(1 - \frac{\zeta a}{\sqrt{x^2+k^2a^2}}\right)\frac{x\,dx}{[x^2+a^2(k^2-\zeta^2)]} .$$

Evaluating (14.14) and (14.15) at $\zeta = k$, we have

$$\left|L^+(k)\right| = \left(\frac{\pi}{4ka}\right)^{\frac{1}{4}} \exp\left[-\frac{ak}{2} - \frac{1}{2\pi}\int_0^\infty \tan^{-1}\left\{\frac{K_0(x)}{\pi I_0(x)}\right\}\right.$$

$$\left.\times\left(1 - \frac{ka}{\sqrt{x^2+k^2a^2}}\right)\frac{dx}{x}\right], \qquad (14.16)$$

and

$$\delta_2(k) = \frac{\pi}{8} + \frac{ak}{\pi}\left(\log\frac{2\pi}{ka\beta} + 1\right) + \sum_{n=1}^{\infty}\left(\frac{ak}{n\pi} - \sin^{-1}\frac{k}{\gamma_n}\right) \quad (14.17)$$

$$- \frac{1}{2\pi}\int_0^{\infty}\log\left\{\sqrt{1+\left(\frac{\pi I_0(x)}{K_0(x)}\right)^2}\, e^{-2x}\right\}\left(1-\frac{ka}{\sqrt{x^2+k^2a^2}}\right)\frac{dx}{x} .$$

Using (14.13) instead of (14.12), we obtain

$$|L^+(\zeta)| = \frac{(k^2-\zeta^2)^{1/8}}{(k+\zeta)^{1/4}}\sqrt{\frac{\pi}{2}\, J_0(\gamma a)\,|H_0^{(1)}(\gamma a)|} \quad (14.18)$$

$$\times \exp\left\{-\frac{a\zeta}{2} + \frac{a\zeta}{2\pi}\int_0^{\infty}\tan^{-1}\left\{\frac{K_0(x)}{\pi I_0(x)}\right\}\frac{x\,dx}{[x^2+a^2(k^2-\zeta^2)]\sqrt{x^2+k^2a^2}}\right\}$$

and

$$\delta_2(\zeta) = \frac{a\zeta}{\pi}\left(\log\frac{2\pi}{\beta ka}+1\right) + \frac{2a\sqrt{k^2-\zeta^2}}{\pi}\left\{\tan^{-1}\frac{\sqrt{k^2-\zeta^2}}{k+\zeta} - \frac{\pi}{4}\right\}$$

$$+ \tfrac{1}{2}\tan^{-1}\left(-\frac{J_0(\gamma a)}{N_0(\gamma a)}\right) + \sum_{n=1}^{\infty}\left[\frac{a\zeta}{n\pi} - \sin^{-1}\left(\frac{\zeta}{\sqrt{\gamma_n^2-\gamma^2}}\right)\right]$$

$$+ \frac{a\zeta}{2\pi}\int_0^{\infty}\log\left\{\sqrt{1+\left(\frac{\pi I_0(x)}{K_0(x)}\right)^2}\, e^{-2x}\right\}\frac{x\,dx}{[x^2+a^2(k^2-\zeta^2)]\sqrt{x^2+k^2a^2}}$$

which may be used for $|\zeta| \neq k$.

15. Verification that $L^+(\zeta)$ yields a solution.

It is now necessary to show that $\lim_{\zeta\to\infty} g(\zeta)$ exists. For this purpose we first consider the asymptotic behavior[7] of $L^+(\zeta)$ for $|\zeta| \to \infty$, $\operatorname{Im}\zeta > 0$. Breaking up the path of integration of (13.5) into two integrals for $|t| < k$ and two integrals for $|t| > k$, and combining, we have

$$L^+(\zeta) = \exp\left[\frac{\zeta}{\pi i}\int_0^k \frac{\log\left\{\frac{\pi i}{2} J_0(a\gamma)H_0^{(1)}(a\gamma)\right\}}{t^2-\zeta^2}\,dt + \frac{\zeta}{\pi i}\int_k^\infty \frac{\log\left\{I_0(a\gamma')K_0(a\gamma')\right\}}{t^2-\zeta^2}\,dt\right] \qquad (15.1)$$

where $\gamma = \sqrt{k^2-t^2}$ and $\gamma' = \sqrt{t^2-k^2}$. Let $x = a\gamma$ in the first integral. Then for $|\zeta| \to \infty$, $\operatorname{Im}\zeta > 0$,

$$\int_0^k \frac{\log\left\{\frac{\pi i}{2} J_0(a\gamma)H_0^{(1)}(a\gamma)\right\}}{t^2-\zeta^2}\,dt \sim - \frac{1}{a\zeta^2}\int_0^{ak} \frac{x\log\left\{\frac{\pi i}{2} J_0(x)H_0(x)\right\}}{\sqrt{a^2k^2-x^2}}\,dx = O(1/\zeta^2) \,. \qquad (15.2)$$

For the second integral in (15.1), we note that the major contribution to the integral will be for t near ζ. The asymptotic form of

[7] Levine and Schwinger, op. cit., p. 406, Ref. (2).

the integrand is $\log \left\{\frac{1}{2at}\right\}/(t^2 - \zeta^2)$. Removing this variation from the integrand, we have

$$\int_k^\infty \frac{\log\left\{I_0(a\gamma')K_0(a\gamma')\right\}}{t^2-\zeta^2}\,dt = \int_k^\infty \frac{\log\left(\frac{1}{2at}\right)}{t^2-\zeta^2}\,dt \qquad (15.3)$$

$$+\int_k^\infty \frac{\log(2atI_0(a\gamma')K_0(a\gamma'))}{t^2-\zeta^2}\,dt\;.$$

For the first integral on the right of (15.3), let $v = -t/i\zeta$. Then for $|\zeta| \to \infty$, $\operatorname{Im}\zeta > 0$,

$$\int_k^\infty \frac{\log\left(\frac{1}{2at}\right)}{t^2-\zeta^2} \sim \frac{1}{-i\zeta}\log\left(\frac{1}{-2ia\zeta}\right)\int_0^\infty \frac{dv}{1+v^2} + \frac{1}{i\zeta}\int_0^\infty \frac{\log\frac{1}{v}}{1+v^2}\,dv$$

$$= -\frac{\pi}{2i\zeta}\log\left(\frac{1}{-2ia\zeta}\right) + O\left(\frac{1}{\zeta}\right)\;. \qquad (15.4)$$

For the second integral on the right of (15.3), let $x = a\gamma'$. Then

$$\left|\int_k^\infty \frac{\log(2at\;I_0(a\gamma')K_0(a\gamma'))}{t^2-\zeta^2}\,dt\right|$$

$$= \left|\int_0^\infty \frac{x\,\log(2\sqrt{x^2+k^2a^2}\;I_0(x)K_0(x))}{(x^2+k^2a^2-a^2\zeta^2)\sqrt{x^2+k^2a^2}}\,dx\right|$$

$$\leq \left|\text{const}\int_0^\infty \frac{dx}{x^2-a^2\zeta^2}\right| = \left|O\left(\frac{1}{\zeta}\right)\right|\;. \qquad (15.5)$$

Collecting results (15.2), (15.4), and (15.5) and substituting them in (15.1), we obtain

$$L^+(\zeta) \sim (-i\zeta)^{-\frac{1}{2}} \quad \text{for } |\zeta| \to \infty, \quad \text{Im } \zeta > 0 .$$

Similarly we may obtain

$$L^-(\zeta) \sim (i\zeta)^{\frac{1}{2}} \quad \text{for } |\zeta| \to \infty, \quad \text{Im } \zeta < 0 .$$

For the asymptotic behavior of $\Phi(\zeta)$ and $\mathcal{E}_z(a, \zeta)$ we use the results obtained in Section 7, since the behavior of $\phi(a,z)$ and $E_z(a,z)$ near $z = 0$ is the same for this case as the previous one. Thus

$$\Phi(\zeta) \sim (-i\zeta)^{\frac{1}{2}} \quad \text{for } |\zeta| \to \infty , \quad \text{Im } \zeta > 0$$

and

$$\mathcal{E}_z(\zeta) \sim 1/(i\zeta)^{\frac{1}{2}} \quad \text{for } |\zeta| \to \infty , \quad \text{Im } \zeta < 0 .$$

Substituting these results into the definition of $g(\zeta)$ given by (13.1), we see that $\lim_{\zeta \to \infty} g(\zeta) = \text{const.}$ We are therefore justified in writing equation (13.2)

16. Integral expressions for $H_\phi(\rho, z)$.

As in the previous problem, H_ϕ is the field component of greater interest than E_z. From equations (8.1), (12.4), and (13.2) and taking the inverse transform, we get

$$H_\phi = -\frac{ck\eta}{2\pi i}\int_{-\infty}^{\infty}\frac{\partial \mathcal{G}(a,\rho,\zeta)}{\partial\rho}\,\frac{e^{i\zeta z}}{L^+(\zeta)(k^2-\zeta^2)}\,d\zeta. \qquad (16.1)$$

Consider the four following regions:

1. $z > 0$, $\rho > a$;
2. $z > 0$, $\rho < a$;
3. $z < 0$, $\rho > a$;

and 4. $z < 0$, $\rho < a$. See Fig. 13.

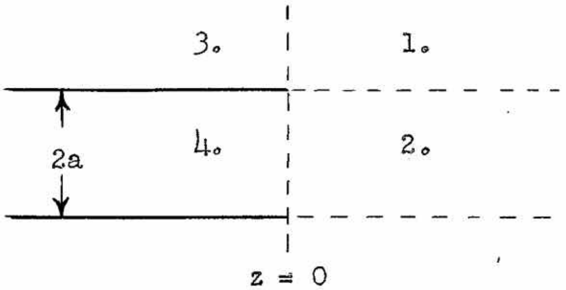

Fig. 13. Regions for the expressions of H_ϕ.

Substituting in the proper form of $\mathcal{G}(a,\rho,\zeta)$ for ρ greater than or less than a, and using $L^+(\zeta)$ for $z > 0$, and $\mathcal{G}(\zeta)\, L^-(\zeta)$ for $z < 0$, equation (16.1) gives for region 1

$$H_\phi = \frac{ck\eta}{4}\int_{-\infty}^{\infty}\frac{J_0(\gamma a)H_1^{(1)}(\gamma\rho)e^{i\zeta z}}{L^+(\zeta)\,\gamma}\,d\zeta; \qquad (16.2)$$

for region 2

$$H_\phi = \frac{ck\eta}{4}\int_{-\infty}^{\infty}\frac{J_1(\gamma\rho)H_0^{(1)}(\gamma a)e^{i\zeta z}}{L^+(\zeta)}\,d\zeta; \qquad (16.3)$$

for region 3

$$H_\phi = \frac{ck\eta}{2\pi i}\int_{-\infty}^{\infty} \frac{H_1^{(1)}(\gamma\rho)e^{i\zeta z}}{H_o^{(1)}(\gamma a)L^-(\zeta)\gamma}\, d\zeta\,; \qquad (16.4)$$

and for region 4

$$H_\phi = \frac{ck\eta}{2\pi i}\int_{-\infty}^{\infty} \frac{J_1(\gamma\rho)e^{i\zeta z}}{J_o(\gamma a)L^-(\zeta)\gamma}\, d\zeta\,. \qquad (16.5)$$

These integrals may be reexpressed by closing the path of integration, in the upper half plane for $z > 0$ and in the lower half plane for $z < 0$, as shown in Fig. 8. After some manipulation, we obtain for region 1

$$H_\phi = -\frac{c\eta e^{ikz}}{2L^+(k)\rho} - \frac{ck\eta}{2}\int_{k}^{\infty} \frac{I_o(a\gamma')I_1(\gamma'\rho)}{L^+(\zeta)\gamma}\, e^{i\zeta z}\, d\zeta\,; \qquad (16.6)$$

for region 2

$$H_\phi = -\frac{ck\eta}{2}\int_{k}^{\infty} \frac{I_o(\gamma' a)I_1(\gamma'\rho)e^{i\zeta z}}{L^+(\zeta)\gamma}\, d\zeta\,; \qquad (16.7)$$

for region 3

$$H_\phi = \frac{ck\eta}{2\pi}\int_{-k}^{-\infty} \frac{\{K_1(\gamma'\rho)I_o(\gamma' a) + I_1(\gamma'\rho)K_o(\gamma' a)\}e^{i\zeta z}}{K_o(\gamma' a)\{K_o(\gamma' a) + i\pi I_o(\gamma' a)\}L^-(\zeta)\gamma}\, d\zeta\,; \qquad (16.8)$$

and for region 4

$$H_\phi = ck\eta \sum_{n=1}^{\infty} \frac{J_1(\gamma_n \rho)e^{K_n z}}{\gamma_n L^-(-iK_n)} \left(\lim_{\zeta \to -iK_n} \frac{(\zeta + iK_n)}{J_o(\gamma a)} \right) . \tag{16.9}$$

where $\gamma_n = \sqrt{k^2 - K_n^2}$ are the roots of $J_o(a\gamma_n) = 0$. Now the nature of the solution may be examined. Here as in the previous case, the solution obtained is incomplete due to the lack of continuity between regions 1 and 2. To obtain continuity it is necessary to add a solution of the homogeneous differential equation (Appendix A, equation (A.11)) satisfying the boundary condition, (10.5). In particular

$$\frac{c\eta e^{ikz}}{2L^+(k)\rho} \tag{16.10}$$

must be added to regions 1 and 3. This may be seen to be the incoming field which was assumed originally. Summarizing results: in regions 1 and 2, H_ϕ is given by (16.3), (16.7), or

$$\frac{c\eta e^{ikz}}{2L^+(k)\rho} + \left\{ \text{either } (16.2) \text{ or } (16.6) \right\} ; \tag{16.11}$$

in region 3, H_ϕ is given by

$$\frac{c\eta e^{ikz}}{2L^+(k)\rho} + \left\{ \text{either } (16.4) \text{ or } (16.8) \right\} ; \tag{16.12}$$

and in region 4, H_ϕ is given by (16.5) or (16.9) .

17. Physical quantities of interest.

Although an exact solution to the problem presented has been found, the problem itself corresponds to no actual physical situation. It is not possible to set up a wave of the form given by (10.1) and (10.2); since such a wave would be transmitting infinite energy along the cylinder. Thus

$$\int \vec{S}_{av} \cdot \vec{n}\, da = 2\pi \int_a^{\infty} \frac{1}{\eta} \frac{A^2}{\rho}\, d\rho \to \infty \,. \qquad (17.1)$$

The situation corresponds to that of a plane wave. The physical quantities which happen to remain finite are the current on the cylinder and the total energy scattered back in the negative z direction. In order to transform the problem into a different problem which may be of greater physical interest, it is necessary to introduce a parameter such as the voltage on the cylinder or a distance L which would replace the infinite upper limit in (17.1). Since the consideration of this problem does not lead to additional information with regard to the coaxial structure, it will not be considered in the present work.

Chapter III

The case of a finite inside cylinder

The results of the previous two chapters are used in the present chapter to obtain an approximate solution for the case of a finite inside cylinder.

18. Statement of the problem.

The metallic boundaries of the region being considered consist of two coaxial half infinite circular cylinders of vanishing thickness, the outside cylinder of radius b extending from $z \rightarrow -\infty$ to $z = 0$ and the inside cylinder of radius a extending from $z \rightarrow -\infty$ to $z = h$ (see Fig. 14).

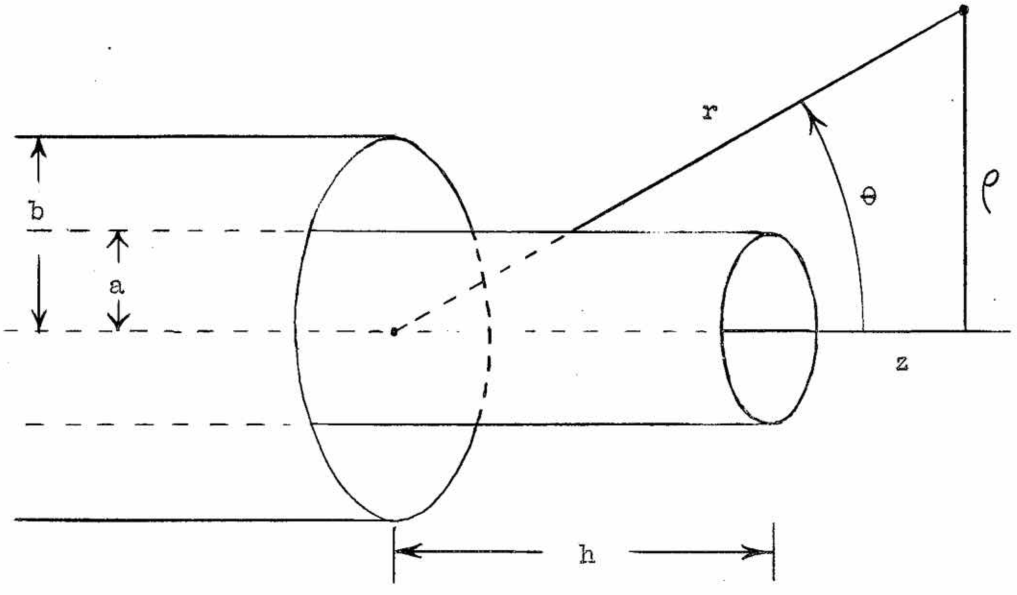

Fig. 14. The coaxial structure for a finite inside cylinder.

A source is assumed to exist in the coaxial region for $z \rightarrow -\infty$ of the form given by (1.1) and (1.2). For a vacuum (or air), ideal metallic boundaries, time harmonic field components, and axial symmetry, Maxwell's equations reduce to (1.3), (1.4) and (1.5). The boundary conditions in this case are

$$E_z = 0 \quad \text{and} \quad \frac{1}{\rho}\frac{\partial}{\partial \rho} H_\phi = 0 \quad \text{for}$$

$$\rho = b, \; z \leq 0 \quad \text{and} \quad \rho = a, \; z \leq h \; . \tag{18.1}$$

The radiation condition (1.8) also applies to the present case outside the cylinder. In order for only the TEM mode to be propagated in the coaxial region we require that $k < \frac{\pi}{b-a}$; and in order for no propagation to occur inside the inside cylinder we require condition (10.6). Thus k is assumed to be chosen small enough to satisfy both conditions simultaneously.

19. Approximate solution for h large.

The following derivation of an approximate solution uses only simple physical notions. An approximate solution almost identical in form but not quite as good was derived by a tedious analytical procedure and served to check the results of this section.

Consider the solution to be composed of two components,

$$H_\phi = H_\phi^{\,0} + H_\phi^{\,1} \tag{19.1}$$

where $H_\phi^{\,0}$ is the field for the case of an infinite inside cylinder and $H_\phi^{\,1}$ is the field scattered from the terminated end of the inside cylinder and therefore corresponds in form to the scattered field for the case of a single half infinite cylinder. This approximation from the nature of the geometry should be best for ρ small, $z < h$. For $\rho \to a$ and z of the order of magnitude of h, we assume $H_\phi{}^o$ may be obtained from (9.14) (which may not be precisely correct inasmuch as (9.14) was derived under the assumption that $\rho \to \infty$), and we get

$$H_\phi{}^o \approx - \frac{c\,\eta \log \frac{b}{a}\, e^{ikz}}{2 \log \frac{2z}{\beta k a^2} M^+(k) \rho} \quad . \tag{19.2}$$

We now relate this field to the incident field for the case of a single half infinite cylinder as given in Chapter II, equation (16.10), i.e.,

$$\frac{c'\eta\, e^{ik(z-h)}}{2L^+(k)\,\rho} \tag{19.3}$$

where a translation of the z axis has been made. Matching these fields, (19.2) and (19.3), at $z = h$, we have

$$\Lambda = -\frac{c'}{c} = \frac{\log \frac{b}{a} L^+(k) e^{ikh}}{\log \frac{2h}{\beta k a^2} M^+(k)} \quad . \tag{19.4}$$

The asymptotic form of $H_\phi{}^1$ for $r' \to \infty$ may be derived from equation (16.2) by the method used in Section 9, to yield

$$H_\phi{}^1 \approx \frac{c\,\eta\,\Lambda\, J_o(ka \sin \theta') e^{ikr'}}{2L^+(k \cos \theta') r' \sin \theta'} \tag{19.5}$$

where the primes denote the coordinates chosen with the origin at

$z = h$. For r and $r' >> h$, we then have

$$H_\phi \approx -\frac{c\eta e^{ikr}}{2r \sin\theta} \left\{ \frac{Z_o(kb \sin\theta)}{\frac{\pi i}{2} H_o^{(1)}(ka \sin\theta) M^+(k \cos\theta)} - \frac{\Lambda J_o(ka \sin\theta)}{L^+(k \cos\theta)} \right\} . \tag{19.6}$$

For $\pi \leq \theta \leq \frac{\pi}{2}$, $L^+(k \cos\theta)$ may be replaced by

$$\frac{\mathcal{G}(k \cos\theta)}{L^+(-k \cos\theta)} \tag{19.7}$$

from (14.9). This approximate solution, (19.6), gives H_ϕ infinite for $\theta = 0$ and therefore fails for the z axis, since from symmetry we know $H_\phi = 0$ on the z axis.

a. Expressions for $|R_2|$ and S_2.

To obtain the reflection coefficient the scattered field H_ϕ^1 given by (19.5) and (19.7) is evaluated for $\rho \to a$ and $z \to 0$,

$$H_\phi^1 \approx \frac{c\eta \Lambda L^+(k) e^{ikh}}{2 \log \frac{2h}{\beta ka^2}} \frac{e^{-ikz}}{\rho} . \tag{19.8}$$

Assuming reciprocity for the reflection at $z = 0$, the reflection coefficient R_1 for the problem of the infinite inside cylinder may be used here. Thus, inside the coaxial region for $z \to -\infty$, we have

$$H_\phi \approx \frac{A}{\rho}\,(e^{ikz} + R_1 e^{-ikz}) + (1 + R_1) H_\phi{}^1 \; . \qquad (19.9)$$

Substituting $H_\phi{}^1$ as given by (19.8) and A as given by (8.11) into (19.9) we obtain

$$R_2 = R_1 \left\{ 1 + (1 + R\;) \Lambda^2 \right\} \qquad (19.10)$$

where R_2 is an approximation of the actual reflection coefficient. Using (9.3), we have

$$|R_2| = |R_1| \Big[1 + (1 + |R_1|^2 - 2|R_1| \cos 2\delta_1)|\Lambda|^4 \qquad (19.11)$$
$$+ 2(1 + |R_1|^2 - 2|R_1| \cos 2\delta_1)^{\frac{1}{2}} |\Lambda|^2 \cos \alpha_o \Big]^{\frac{1}{2}}$$

where

$$\alpha_o = 2kh + 2\delta_2(k) - 2\delta_1(k) - \tan^{-1}\left(\frac{|R_1| \sin 2\delta_1}{1 - |R_1| \cos 2\delta_1} \right) \; . \qquad (19.12)$$

The distance s_2 is given by the phase of (19.10), or

$$ks_2 = ks_1 + \tfrac{1}{2} \tan^{-1} \left\{ \frac{\sin \alpha_o}{\cos \alpha_o + 1 / |\Lambda|^2 (1 + |R_1|^2 - 2|R_1| \cos 2\delta_1)^{\frac{1}{2}}} \right\} \qquad (19.13)$$

b. Gain function, $\mathcal{G}_2(\theta)$.

To obtain the gain function reexpress equation (9.11) by letting

$$H_\phi{}^{o} = f_o(\theta)\,\frac{e^{ikr}}{r}$$

and

$$H_\phi{}^{1} = -\,f_1(\theta)\,\frac{e^{ikr}}{r}\,.$$

which gives

$$\mathcal{G}_2(\theta) = \frac{1}{2\eta P_{t2}}\left\{|f_o(\theta|^2 + |f_1(\theta)|^2\right. \tag{19.15}$$

$$\left. - 2|f_o(\theta)|\,|f_1(\theta|\,\cos\left(\arg f_o(\theta) - \arg f_1(\theta)\right)\right\}.$$

The total power radiated, P_{t2}, as obtained from (8.12), (9.5), and (19.9) is

$$P_{t2} = \frac{\pi\eta c^2}{4}\,\frac{(1-|R_2|^2)}{|R_1|}\,. \tag{19.16}$$

Noting the expressions for $f_o(\theta)$ and $f_1(\theta)$ as given by (19.6), we obtain

$$\mathcal{G}_2(\theta) = \frac{|R_1|}{2\pi(1-|R_2|^2)\sin^2\theta}\left\{\begin{matrix} X^2(\theta) + Y^2(\theta) \\ -2X(\theta)Y(\theta)\cos\alpha_1 \end{matrix}\right\} \tag{19.17}$$

where

$$X(\theta) = \frac{Z_0(kb \sin \theta)}{\frac{\pi}{2} \left|H_o^{(1)}(ka \sin \theta)\right| \left|M^+(k \cos \theta)\right|} , \qquad (19.18)$$

$$Y(\theta) = \frac{|\Lambda| J_o(ka \sin \theta)}{\left|L^+(k \cos \theta)\right|} , \qquad (19.19)$$

and

$$\alpha_1 = kh + [\delta_2(k) - \delta_2(k \cos \theta)] - [\delta_1(k) - \delta_1(k \cos \theta)]$$

$$+ \tan^{-1}\left(- \frac{J_o(ka \sin \theta)}{N_o(ka \sin \theta)}\right) . \qquad (19.20)$$

It is not possible to normalize this approximation of the gain function since $\sin \theta \, \mathcal{G}_2(\theta)$ is not integrable for $\theta \rightarrow 0$; althouth it is integrable for $\theta \rightarrow \pi$. It may be readily seen that all of these quantities reduce to the case of an infinite inside cylinder when $h \rightarrow \infty$ since $\Lambda \sim \frac{1}{\log h} \rightarrow 0$.

Chapter IV

Results

The mathematical results obtained in the previous chapters were evaluated numerically for certain choices of the parameters involved. The numerical results along with a few approximate formulae are contained in the present chapter. Most of the graphs have been drawn using some points which have been obtained by some process of interpolation. This was considered a necessary time saver, because of the complexity of the expressions involved. All numbers which are presented here are considered to have at most an error in the last place unless otherwise indicated.

20. Procedure for obtaining $M^+(k \cos \theta)$ and $L^+(k \cos \theta)$.

The integrals which appear in (6.22), (6.23), (14.14), and (14.15) for $\zeta = k \cos \theta$ were evaluated numerically by making the substitution

$$x = \frac{y}{1-y} \qquad (20.1)$$

which changes the limits of integration from 0 and ∞ to 0 and 1 respectively. Forty points in this interval 0 to 1 were used except in the cases $\theta = 8^\circ$, 2°, and $.5^\circ$ where an additional ten to twenty points were used for y near 0. All calculations were done using 8 decimal places. This technique is assumed to give 4 place accuracy. The series in (6.33), hence forth designated as S_1, was computed by an approximate formula which gives an accuracy of about 3 significant figures. Let

$$S_1\left(\frac{b}{a}, k(b-a), \theta\right) = \sum_{n=1}^{\infty} f_1(n)$$

where

$$f_1(n) = \frac{k(b-a)\cos\theta}{n\pi} - \sin^{-1}\left(\frac{\cos\theta}{\sqrt{\left(\frac{x_n\left(\frac{b}{a}-1\right)}{k(b-a)}\right)^2 - \sin^2\theta}}\right) \quad (20.2)$$

and

$$J_o(x_n)N_o\left(\frac{b}{a}x_n\right) - J_o\left(\frac{b}{a}x_n\right)N_o(x_n) = 0 ,$$

then the approximate formula used may be written

$$S_1 \approx \sum_{n=1}^{5} f_1(n) + \frac{1}{6^3}\left\{2\cdot5^3f_1(5) - 4^3f_1(4)\right\} \quad (20.3)$$

$$+\left\{3\cdot5^3f_1(5) - 2\cdot4^3f_1(4)\right\}\left(\zeta(3) - \sum_{n=1}^{6}\frac{1}{n^3}\right)$$

where $\zeta(3)$ is the Riemann zeta function for the argument 3. Formula (20.3) may be derived by expanding $f_1(n)$ as given by (20.2) in inverse powers of n and summing the leading term of $1/n^3$. The values[1] of S_1 as given by (20.3) are given in a supplementary table at the end of this chapter.

Similarly the series in (14.15), henceforth designated as S_2, defined by

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[1]Cf. N. Marcuritz, "Wave guide Handbook" (McGraw-Hill Co. New York, 1951) Appendix, Ref. (10).

$$S_2(ka, \theta) = \sum_{n=1}^{\infty} f_2(n)$$

where

$$f_2(n) = \frac{ka \cos \theta}{n\pi} \sin^{-1} \left(\frac{\cos \theta}{\sqrt{\left(\frac{x_n}{ka}\right)^2 - \sin^2\theta}} \right) \qquad (20.4)$$

and

$$J_0(x_n) = 0 ,$$

were evaluated approximately by:

$$S_2 \approx \sum_{n=1}^{6} f_2(n) + \frac{1}{7^3} \left\{2\cdot 6^3 f_2(6) - 5^3 f_2(5)\right\}$$

$$+ \left\{3\cdot 6^3 f_2(6) - 5^3 f_2(5)\right\} \left(\zeta(3) - \sum_{n=1}^{7} \frac{1}{n^3}\right)$$

$$+ \frac{ka \cos \theta}{4\pi} \left[8 \left(\zeta(3) - \sum_{n=1}^{7} \frac{1}{n^3}\right) - \left(\zeta(2) - \sum_{n=1}^{7} \frac{1}{n^2}\right)\right] .$$

This formula may be obtained by expanding (20.4) in inverse powers of n and summing the terms in $1/n^2$ and $1/n^3$. The values of S_2 as given by (20.5) are given in the supplementary table at the end of this chapter.

The numerical results for $|M^+(k \cos \theta)|$, $\delta_1(k \cos \theta)$, $|L^+(k \cos \theta)|$, and $\delta_2(k \cos \theta)$ are listed in Table I, in case they may be of value for future approximations of the case of a finite inside cylinder. A numerical check on the integrations was afforded by the fact that there is a relation between $M^+(k\cos\theta)$ and $L^+(k\cos\theta)$.

Table I. Values† of $M^+(k\cos\theta)$ and $L^+(k\cos\theta)$.

$\frac{b}{a}$	$k(b-a)$	θ in degrees	$\lvert M^+(k\cos\theta)\rvert$	$\delta_1(k\cos\theta)$	$\lvert L^+(k\cos\theta)\rvert$	$\delta_2(k\cos\theta)$
1.25	0	0	.47238	0		
	.30		.38537	.3355		
	1.20		.24524	.7831		
	3.00		.09969	.7605		
2.00	0		.83255	0	∞	0
	.12				1.36692	.4261
	.15		.67477	.2282	1.29349	.4627
	.30		.61703	.3563	1.04835	.6020
	.48				.86695	.7185
	.60		.52504	.5466	.77696	.7785
		.5	.525	.5465*	.777	.7784*
		2.0	.52514	.5460*	.77711	.7781*
		8.0	.52628	.5440*	.77944	.7770*
		15.0	.52945	.5416	.78573	.7757
		30.0	.54280	.5261	.81278	.7670
		45.0	.56549	.4990	.86040	.7506
		60.0	.59796	.4583	.93250	.7237
		75.0	.64051	.4017	1.03470	.6824
		90.0	.69283	.3269	1.17441	.6223
		105.0	.75358	.2324	1.36109	.5387
		120.0	.82004	.1179	1.60757	.4278
		135.0	.88836	-.0149	1.93543	.2871
		150.0	.95511	-.1621	2.39548	.1136
		165.0	1.02277	-.3197	3.17060	-.1032
		172.0	1.06247	-.3967*	3.88044	-.2330*
		178.0	1.12331	-.4761*	5.5071	-.4024*
		179.5	1.164	-.5070*	7.20	-.4937*
		180.0	1.32019	-.5466	∞	-.7785
	1.20	0	.38618	.7923*	.48998	.9487
		.5	.3862	.7923*	.4900	.9488*
		2.0	.38625	.7921*	.49011	.9491*
		8.0	.38754	.7912*	.49170	.9500*
		15.0	.39099	.7903	.49614	.9511
		30.0	.40601	.7833	.51571	.9578
		45.0	.43301	.7688	.55233	.9675
		60.0	.47506	.7421	.61322	.9760
		75.0	.53646	.6955	.71085	.9749
		90.0	.62195	.6191	.86444	.9492
		105.0	.73476	.5027	1.10017	.8803

Table I. (Continued)

$\frac{b}{a}$	k(b-a)	Θ in degrees	$\|M^+(k\cos\Theta)\|$	$\delta_1(k\cos\Theta)$	$\|L^+(k\cos\Theta)\|$	$\delta_2(k\cos\Theta)$
2.00	1.20	120.0	.87335	.3388	1.44979	.7516
		135.0	1.02821	.1261	1.95183	.5543
		150.0	1.18226	- .1295	2.67446	.2868
		165.0	1.32228	- .4160	3.8674	-.0566
		172.0	1.39142	- .5576*	4.9420	-.2588
		178.0	1.49089	- .6932*	7.4293	-.5034
		179.5	1.5574	- .7407*	10.06	-.6247
		180.0	1.79490	- .7923	∞	-.9487
	2.40	0	.21122	.9365	.22727††	
		.5	.211	.9367*		
		2.0	.21125	.9372*		
		8.0	.21170	.9394*		
		15.0	.21293	.9419		
		30.0	.21840	.9595		
		45.0	.22897	.9938		
		60.0	.24847	1.0515		
		75.0	.28725	1.1339		
		90.0	.36935	1.2116		
		105.0	.53545	1.2082		
		120.0	.82799	1.0530		
		135.0	1.25871	.7336		
		150.0	1.77114	.2787		
		165.0	2.2412	- .2677		
		172.0	2.4274	- .5424*		
		178.0	2.6465	- .7897*		
		179.5	2.79	- .8668*		
		180.0	3.2816	- .9365		
	3.00	0	.15639	.7486		
3.50	0		1.11927	0		
	.30		.75306	.3820		
	1.20		.45888	.8080		
	3.00		.18380	.7121		

† The values of $|L^+(k\cos\Theta)|$ and $\delta_2(k\cos\Theta)$ are listed for $\frac{b}{a} = 2.00$ in order for the column headed k(b-a) to be identical to ka.

*Interpolated values

††Calculated from (14.16), but is not applicable to problems

21. Results for the case of an infinite inside cylinder.

The numerical results for Chapter I for the physical quantities of interest are given in this section.

a. Output impedance, G_1/Y_0.

The output impedance,

$$G_1/Y_0 = \frac{1 - |R_1|}{1 + |R_1|}, \tag{21.1}$$

given by equation (9.4) and (9.8), where

$$Y_0 = \frac{2\pi\eta}{\log \frac{b}{a}},$$

is plotted in Graph I as a function of $k(b-a)$ for three choices of $\frac{b}{a}$. Numerical values are given in Table II for $\frac{h}{a} = \infty$.

A further check on the values of G_1/Y_0 for the case $\frac{b}{a} = 2$ was obtained by evaluating $|M^+(k)|$ using the first method, equation (5.6). The actual expression used was derived by integrating by parts to get

$$|M^+(k)| = \sqrt{\log \frac{b}{a}} \exp\left[-\frac{2}{\pi^2}\int_0^k \log\left(\frac{k+\sqrt{k^2-x^2}}{x}\right)\left\{\frac{1}{|H_0^{(1)}(bx)|^2} - \frac{1}{|H_0^{(1)}(ax)|^2}\right\}\frac{dx}{x}\right], \tag{21.2}$$

subtracting out the integrable infinity at x = 0, and further smoothing the integrand by subtracting the asymptotic form of the Hankel functions to finally obtain

$$|M^+(k)| = \sqrt{\log \frac{b}{a}}\ \exp\left[-k(b-a)\left[\frac{1}{2}-\frac{1}{\pi}-\frac{1}{\pi}\log 2\right]-\log\frac{b}{a}\right.$$

$$+\frac{1}{\pi}\log\left(\frac{1}{\beta ka}\right)\tan^{-1}\left(\frac{2}{\pi}\log\frac{2}{\beta ka}\right)$$

$$-\frac{1}{\pi}\log\left(\frac{1}{\beta kb}\right)\tan^{-1}\left(\frac{2}{\pi}\log\frac{2}{\beta kb}\right)$$

$$-\frac{2}{\pi^2}\int_0^{k(b-a)}\left[\log\left(\frac{k(b-a)+\sqrt{k^2(b-a)^2-x^2}}{x}\right)\left\{\frac{1}{\left|H_o^{(1)}\left(\frac{bx}{b-a}\right)\right|^2}\right.\right.$$

$$\left.-\frac{1}{\left|H_o^{(1)}\left(\frac{ax}{b-a}\right)\right|^2}-\frac{\pi}{2}x\right\}$$

$$-\log\left(\frac{2k(b-a)}{x}\right)\left\{\frac{1}{1+\frac{4}{\pi^2}\log^2\frac{2(b-a)}{\beta bx}}\right.$$

$$\left.\left.\left.-\frac{1}{1+\frac{4}{\pi^2}\log^2\frac{2(b-a)}{\beta ax}}-\frac{\pi}{2}x\right\}\right]\frac{dx}{x}\ . \qquad (21.3)$$

The resulting numerical values of G_1/Y_0 are also listed in Table II. The integration in (21.3) was carried out numerically by choosing 24 to 40 points. Six rather than eight places were used in this computation.

An approximation for k large or $k(b-a) > 1$ may be obtained by considering $|M(k)|$ as given by (6.24) and expanding the radical in the integrand for ka and kb large to get

$$|M^+(k)| = \left(\frac{a}{b}\right)^{\frac{1}{4}} \sqrt{\log \frac{b}{a}}\; e^{-\frac{k(b-a)}{2}} \tag{21.4}$$

$$\times \exp\left[-\frac{1}{4\pi k^2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\int_0^\infty x \tan^{-1}\left\{\frac{K_0(x)}{\pi I_0(x)}\right\} dx + O\left(\frac{1}{(ka)^4}\right)\right].$$

Using (14.13) and (14.12) to evaluate P(0) and expanding the Hankel function for a large argument, we find

$$1 + \frac{1}{8iak} + O\left(\frac{1}{(ak)^2}\right) \tag{21.5}$$

$$= \exp\left[\frac{1}{\pi i}\int_0^\infty \log\left\{\left(\frac{\pi I_0(x)}{K_0(x)} - i\right) e^{-2x}\right\}\frac{x dx}{x^2+k^2a^2} + O\left(\frac{1}{(ka)^4}\right)\right].$$

Taking the absolute value of (21.5) and letting $k \to \infty$, we have

$$\int_0^\infty x \tan^{-1}\left(\frac{K_0(x)}{\pi I_0(x)}\right) dx = -\frac{\pi}{128}\ ; \tag{21.6}$$

and substituting this result into (21.4), we obtain

$$|R_1| = \sqrt{\frac{a}{b}}\, e^{-k(b-a)} \left[1 + \frac{1}{256k^2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) + 0\left(\frac{1}{(ka)^4}\right)\right] , \qquad (21.7)$$

or approximately

$$|R_1| \approx \sqrt{\frac{a}{b}}\, e^{-k(b-a)} .$$

For $k(b-a) \geq .3$, (21.8) is accurate to about $3\left(\frac{b}{a} - 1\right)^2$ per cent and for $k(b-a) \geq 1.2$, (21.8) is accurate to about $\left(\frac{b}{a} - 1\right)^2$ per cent. The approximate expression for G_1/Y_o corresponding to (21.8) is

$$\frac{G_1}{Y_o} \approx \tanh\left\{\frac{k(b-a)}{2} + \frac{1}{4}\log\frac{b}{a}\right\} . \qquad (21.9)$$

For k small, (21.2) is considered; since the greatest contribution is near x = 0, we make the substitution

$$\log\left(\frac{k + \sqrt{k^2 - x^2}}{x}\right) \approx \log\frac{2k}{x}$$

and replace the Hankel function by its expression for a small argument. Integrating we obtain

$$|M^+(k)| = \sqrt{\frac{a}{b}} \left(\frac{1 + \frac{4}{\pi^2} \log^2 \frac{2}{\beta kb}}{1 + \frac{4}{\pi^2} \log^2 \frac{2}{\beta ka}} \right)^{\frac{1}{4}} \left[1 + O(k^2(b-a)^2 \log k(b-a) \right]$$

$$\times \exp\left\{ \frac{1}{\pi} (\log \beta kb) \tan^{-1} \left(\frac{2}{\pi} \log \frac{2}{\beta kb} \right) \right.$$

$$\left. - \frac{1}{\pi} (\log \beta ka) \tan^{-1} \left(\frac{2}{\pi} \log \frac{2}{\beta ka} \right) \right\} \tag{21.10}$$

for $k < 1$.

b. <u>Distance s_1.</u>

The distance s_1 as given by (6.25) is plotted in Graph II as the ratio $\frac{s_1}{(b-a)}$ for three values of $\frac{b}{a}$. The numerical values are given in Table II for $\frac{h}{a} = \infty$.

For $k(b-a)$ large, $k(b-a) > 1$, we consider the imaginary part of (21.6) and make the substitution

$$\frac{x}{x^2 + k^2a^2} = \frac{2}{x} \left(1 - \frac{ka}{\sqrt{x^2 + k^2a^2}} \right) + O\left\{ \frac{x^3}{(ka)^6} \right\}$$

to get

$$\frac{2}{\pi}\int_0^\infty \log\left(\sqrt{1+\left(\frac{\pi I_o(x)}{K_o(x)}\right)^2}\; e^{-2x}\right)\left(1-\frac{ka}{\sqrt{x^2-k^2a^2}}\right)\frac{dx}{x}$$

$$= \frac{1}{8\ ka} + 0\left(\frac{1}{(ka)^2}\right). \qquad (21.11)$$

Then from equation (6.25) we find

$$ks_1 = \frac{k(b-a)}{\pi}\log\frac{2\pi e}{\beta k(b-a)} + S_1(k) \qquad (21.12)$$

$$+ \frac{1}{32k}\left(\frac{1}{a}-\frac{1}{b}\right) + 0\left(\frac{1}{(ka)^2}\right)$$

for $k(b-a) > 1$.

c. Gain function, $\mathcal{G}_1(\theta)$.

The gain function $\mathcal{G}_1(\theta)$ as given by (9.17) is plotted in Graph III for the case $\frac{b}{a} = 2$ and three choices of the frequency $k(b-a)$. The numerical results are given in Table III for $\frac{h}{a} = \infty$.

From (6.22) we may obtain the approximation

$$\left|M^+(k\cos\theta)\right| \approx \left(\frac{1+\frac{4}{\pi^2}\log^2\frac{2}{\beta kb\theta}}{1+\frac{4}{\pi^2}\log^2\frac{2}{\beta ka\theta}}\right)^{\frac{1}{4}} \left|M^+(k)\right| \qquad (21.13)$$

for $\theta \to 0$. Substituting this result into (9.17), we find

$$\mathcal{G}_1(\theta) \approx$$

$$\frac{2}{\pi^3} \frac{\log \frac{b}{a}}{(1-|R_1|^2)\theta^2 \left(1+\frac{4}{\pi^2}\log^2\frac{2}{\beta kb\theta}\right)^{\frac{1}{2}}\left(1+\frac{4}{\pi^2}\log^2\frac{2}{\beta ka\theta}\right)^{\frac{1}{2}}} \quad (21.14)$$

for $\theta \to 0$. Similarly for $\theta \to \pi$, we find

$$|M^+(k\cos\theta| \approx \left(\frac{1+\frac{4}{\pi^2}\log^2\frac{2}{\beta kb(\pi-\theta)}}{1+\frac{4}{\pi^2}\log^2\frac{2}{\beta ka(\pi-\theta)}}\right)^{\frac{1}{4}} \frac{\log\frac{b}{a}}{|M^+(k)|};$$

so that

$$\mathcal{G}_1(\theta) \approx$$

$$\frac{2}{\pi^3} \frac{|R_1|^2 \log \frac{b}{a}}{(1-|R_1|^2)(\pi-\theta)^2 \left(1+\frac{4}{\pi^2}\log^2\frac{2}{\beta kb(\pi-\theta)}\right)^{\frac{1}{2}}\left(1+\frac{4}{\pi^2}\log^2\frac{2}{\beta ka(\pi-\theta)}\right)^{\frac{1}{2}}} \quad (21.15)$$

for $\theta \to \pi$.

22. <u>Results for the case of a finite inside cylinder.</u>

Since the numerical results for this problem are obtained from an approximate solution, a proper evaluation will depend upon agreement with experiment.

a. Output inpedance, G_2/Y_o.

The output impedance as given by (19.11) and (21.1), $|R_2|$ replacing $|R_1|$ is plotted in Graph IV as a function of ka for three choices of the parameter $\frac{b}{a}$ for $\frac{h}{a} = 10$, in Graph V as a function of ka for three choices of the parameter $\frac{h}{a}$ for $\frac{b}{a} = 2$, and in Graph VI as a function of $\frac{h}{a}$ for $\frac{b}{a} = 2$ and $k(b-a) = 1.2$. Numerical values are given in Table II for $\frac{h}{a} \neq \infty$.

The procedure used in Section 21 to obtain an approximation for $|M^+(k)|$ for $k > 1$ may also be used here for $|L^+(k)|$ to give

$$L^+(k) \approx \left(\frac{\pi}{4ak}\right)^{\frac{1}{4}} e^{-\frac{ak}{2}}\left(1 + \frac{1}{512(ka)^2}\right) \qquad (22.1)$$

or simply

$$L^+(k) \approx \left(\frac{\pi}{4ak}\right)^{\frac{1}{4}} e^{-\frac{ak}{2}} .$$

Substitute (22.2) and the first part of (21.4) into (19.4), we obtain

$$|\Lambda| \approx \frac{\sqrt{\log \frac{b}{a}}}{\log \frac{2h}{\beta\, ka^2}} \left(\frac{\pi}{4kb}\right)^{\frac{1}{4}} e^{-\frac{k(2a-b)}{2}} \qquad (22.3)$$

Neglecting the squared terms of $|R_1|$ and $|\Lambda|^2$ in (19.11), we obtain

$$|R_2| \approx \sqrt{\frac{a}{b}}\, e^{-k(b-a)} + \sqrt{\frac{a\pi}{4b^2k}}\, \frac{\log \frac{b}{a}}{\log^2 \frac{h}{\beta ka^2}}\, e^{-ak} \cos \alpha_0 \,. \qquad (22.4)$$

The phase angle α_0 which is given by (19.12) cannot be conveniently simplified. The expression (22.4) gives $|R_2|$ as $|R_1|$ plus a correction term which clearly goes to zero as $h \to \infty$.

b. Distance s_2.

The distance s_2, as given by (19.13) is plotted as $\frac{s_2}{(b-a)}$ in Graph VII as a function of ka for three choices of the parameter $\frac{b}{a}$ for $\frac{h}{a} = 10$ in Graph VIII as a function of ka for three choices of the parameter $\frac{h}{a}$ for $\frac{b}{a} = 2$, and in Graph IX as a function of $\frac{h}{a}$ for $\frac{b}{a} = 2$ and $k(b-a) = 1.2$. Numerical values are also given in Table II for $\frac{h}{a} \neq \infty$.

As in Section 21b, we find, using equation (14.17) that

$$\delta_2(k) = \frac{\pi}{8} + \frac{ka}{\pi} \log \frac{2\pi e}{\beta ka} + S_2(k) - \frac{1}{32ka} + 0\left(\frac{1}{(ka)^2}\right) \qquad (22.5)$$

for $ka > 1$. Expanding the arctangent in (19.13), we obtain

$$ks_2 \approx ks_1 + \tfrac{1}{2}|\Lambda|^2 \,(1 - |R_1| \cos 2\,\delta_1) \sin \alpha_0 \qquad (22.6)$$

for $ka > 1$.

Expression (22.6) gives ks_2 as ks_1 plus a correction term which becomes zero for $h \to \infty$

c. Gain function $\mathcal{G}_2(\theta)$.

The gain function $\mathcal{G}_2(\theta)$ is plotted in Graph X as a function of θ for $\frac{b}{a} = 2$ and $\frac{h}{a} = 10$ for two choices of the parameter $k(b-a)$; and in Graph XI as a function of θ for $\frac{b}{a} = 2$, and $k(b-a) = 1.2$ for three choices of the parameter $\frac{h}{a}$. The numerical values are also given in Table III.

For $\theta \to 0$, we let

$$|L^+(k \cos \theta)| \approx |L^+(k)| \; . \tag{22.7}$$

Then from (19.19) and (19.5),

$$Y(\theta) \approx \frac{\log \frac{b}{a}}{\log \frac{2h}{\beta ka^2} \, |M^+(k)|} \; ; \tag{22.8}$$

and neglecting $Y^2(\theta)$, we obtain

$$\mathcal{G}_2(\theta) \approx \left(\frac{1 - |R_1|^2}{1 - |R_2|^2} \right) \mathcal{G}_1(\theta) \tag{22.9}$$

$$- \frac{2 \log \frac{b}{a} \cos \alpha_1}{\pi^2 (1 - |R_2|^2) \log \frac{2h}{\beta ka^2} \, \theta^2 \left\{ 1 + \frac{4}{\pi^2} \log^2 \frac{2}{\beta ka\theta} \right\}^{\frac{1}{4}} \left\{ \cdot + \frac{4}{\pi^2} \log^2 \frac{2}{\beta kb\theta} \right\}^{\frac{1}{4}}}$$

for $\theta \to 0$ where $\mathcal{G}_1(\theta)$ is given by (21.14). From symmetry we know that the field must be zero along the z axis; so that approximation (22.9) may not be adequate. If as in Section 19 we consider H_ϕ to be given by:

$$H_\phi = -\frac{c\,\eta \log \frac{b}{a}}{2\log \frac{2h}{\beta ka^2} L^+(k) r \sin\theta}\left[e^{ikr\cos\theta} - \frac{L^+(k)}{L^+(k\cos\theta)} e^{ikr}\right] \quad (22.1))$$

for the region of the z axis, $z \approx h$, we find that

$$\mathcal{G}_2(\theta) \approx \frac{h^2 \log \frac{b}{a}}{8\pi(1-|R_2|^2)\log^2 \frac{2h}{\beta ka^2}}\,\theta^2 \quad (22.11)$$

for $\theta \to 0$, which seems to be a better result qualitatively. Considering the definition of $L^+(\zeta)/L^-(\zeta)$ and (13.5), we find

$$|L^+(k\cos\theta| \approx \frac{\frac{\pi}{2}\left(1+\frac{4}{\pi^2}\log^2\frac{2}{\beta ka(\pi-\theta)}\right)^{\frac{1}{2}}}{|L^+(k)|} \quad (22.12)$$

for $\theta \to \pi$. Substituting (22.12) into (19.17) we obtain

$$\mathcal{G}_2(\theta) \approx \left(\frac{1-|R_1|^2}{1-|R_2|^2}\right)\mathcal{G}_1(\theta)$$

$$-\frac{4\log\frac{b}{a} L^+(k)\left(1+\frac{4}{\pi^2}\log^2\frac{2}{\beta ka(\pi-\theta)}\right)^{\frac{1}{4}}\cos\alpha_1}{\pi^3(1-|R_2|^2)\log\frac{2h}{\beta ka^2}\left(1+\frac{4}{\pi^2}\log^2\frac{2}{\beta kb(\pi-\theta)}\right)(\pi-\theta)^2} \quad (22.13)$$

for $\theta \to \pi$ where (21.15) is used for $\mathcal{G}_1(\theta)$. This approximation should be adequate for $\frac{h}{a}$ large.

For $\theta = 90^{\circ}$, $\mathcal{G}_1(\theta)$ and $\mathcal{G}_2(\theta)$ may be expressed more simply by noting that

$$M^{+}(0) = \sqrt{\frac{H_o^{(1)}(kb)}{H_o^{(1)}(ka)}}\, Z_o(kb)$$

and

$$L^{+}(0) = \sqrt{\frac{\pi}{2}\, J_o(ka) H_o^{(1)}(ka)}$$

from equations (6.22) and (14.18).

Table II. Values of G/Y_0 and $\frac{s}{(b-a)}$.

$\frac{b}{a}$	$\frac{h}{a}$	ka	G/Y_0	$\frac{s}{b-a}$
1.25	∞	0	0	∞
		1.200	.20081	1.1185
		4.800	.57540	.6526
		12.000	.91473	.2535
	10.0	0	0	∞
		1.200	.20115	1.1185
2.00	∞	0	0	∞
		.025	.10964*	
		.120	.18843*	
		.150	.20709	1.5215
		.300	.29093	1.1877
			.29094*	
		.600	.43092	.9111
			.43094*	
		1.200	.64588	.6603
			.64590*	
		2.400	.87905	.3902
		3.000	.93183	.2495
			.93184*	
	15.0	1.200	.63638	.6115
	14.5		.67434	.6296
	14.0		.67468	.6911
	13.5		.63606	.7089
	13.0		.61002	.6674
	12.5		.62532	.6160
	12.0		.66924	.6124
	11.5		.68555	.6807
	$\frac{\pi}{2.4}+10.0$		.66511	
	11.0		.64548	.7211
	10.5		.60617	.6836
	10.0	0	0	∞
		.15	.22930	1.5682
		.30	.26030	1.1497
		.60	.39267	.8705
		1.20	.62005	.6081
	9.5		.67326	.5998
	9.0		.69571	.6852
	$-\frac{\pi}{2.4}+10.0$		.66943	
	8.5		.64528	.7372

*Values obtained by using the first method, equation (21.3).

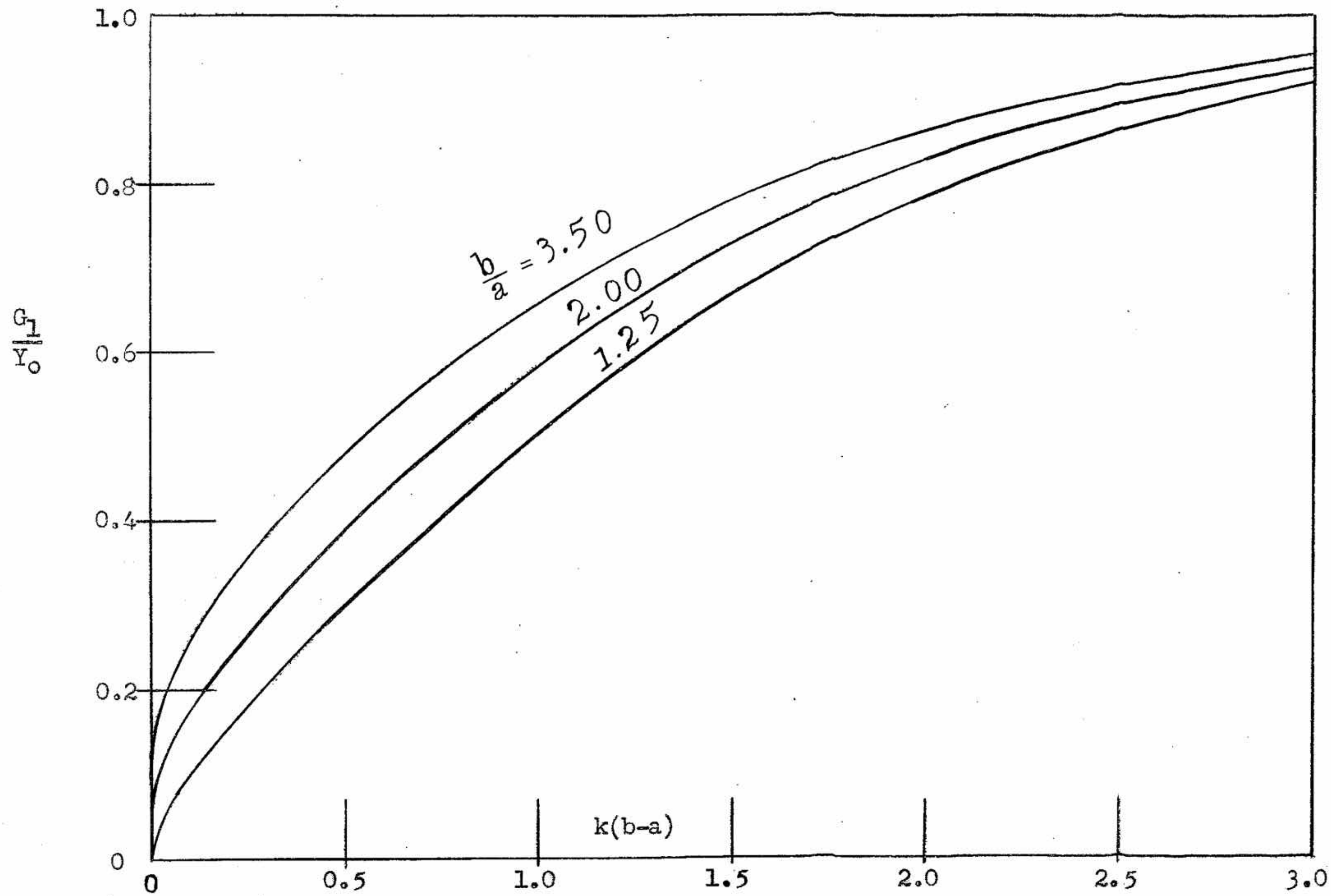
Table II. Continued.

$\frac{b}{a}$	$\frac{h}{a}$	ka	G/Y_o	$\frac{s}{b-a}$
2.00	8.0	1.20	.59526	.6905
	7.5		.60001	.6092
	7.0		.66334	.5670
	6.5		.72001	.6611
	6.0		.66396	.7722
	5.5		.58007	.7274
	5.0	0	0	∞
		.15	.19918	1.7381
		.30	.34250	1.2291
		.60	.36001	.8813
		1.20	.56118	.6125
	4.5		.63519	.5030
	4.0		.77963	.5695
	3.5		.70677	.9095
	3.0		.50507	.8385
	2.5		.37967	.6207
3.50	∞	0	0	∞
		.12	.37677	1.2733
		.48	.71220	.6733
		1.20	.94749	.2374
	10	0	0	∞
		.12	.40818	1.5713
		.48	.86137	.7650
		1.20	.84744	.0180

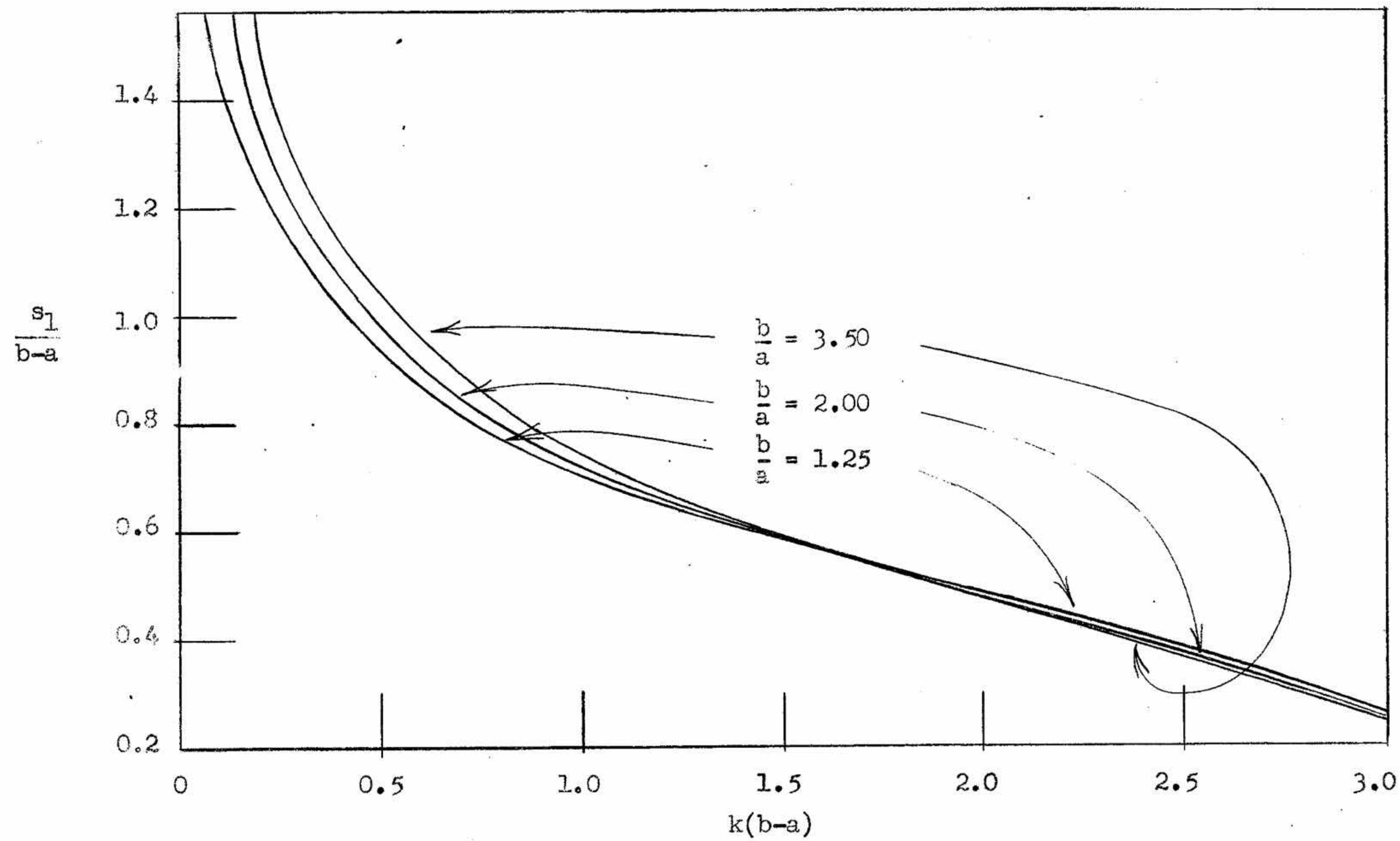
Table III. Values of $4\pi\mathcal{G}(\theta)$ for $\frac{b}{a} = 2$.

θ in degrees	$k(b-a) = 0.6$, $\frac{h}{a} = \infty$	1.2, ∞	2.4, ∞	0.6, 10
0	∞	∞	∞	∞
2	74.652	89.852	124.39	17.22
8	9.3066	12.3285	18.834	.6718
15	3.8636	5.3020	8.0728	.5875
30	1.5255	2.0348	2.5232	.5285
45	.89016	1.06421	.89371	.4380
60	.60638	.62125	.31480	.3587
75	.45473	.39273	.11789	.2941
90	.37023	.27150	.054163	.2458
105	.32851	.20935	.033926	.2132
120	.32242	.18382	.028347	.1969
135	.36069	.18874	.029573	.2007
150	.49273	.23998	.038368	.2435
165	1.03535	.46359	.072869	.4432
172	2.2834	.95637	.14326	.8975
178	16.315	6.0307	.79251	5.698
180	∞	∞	∞	∞

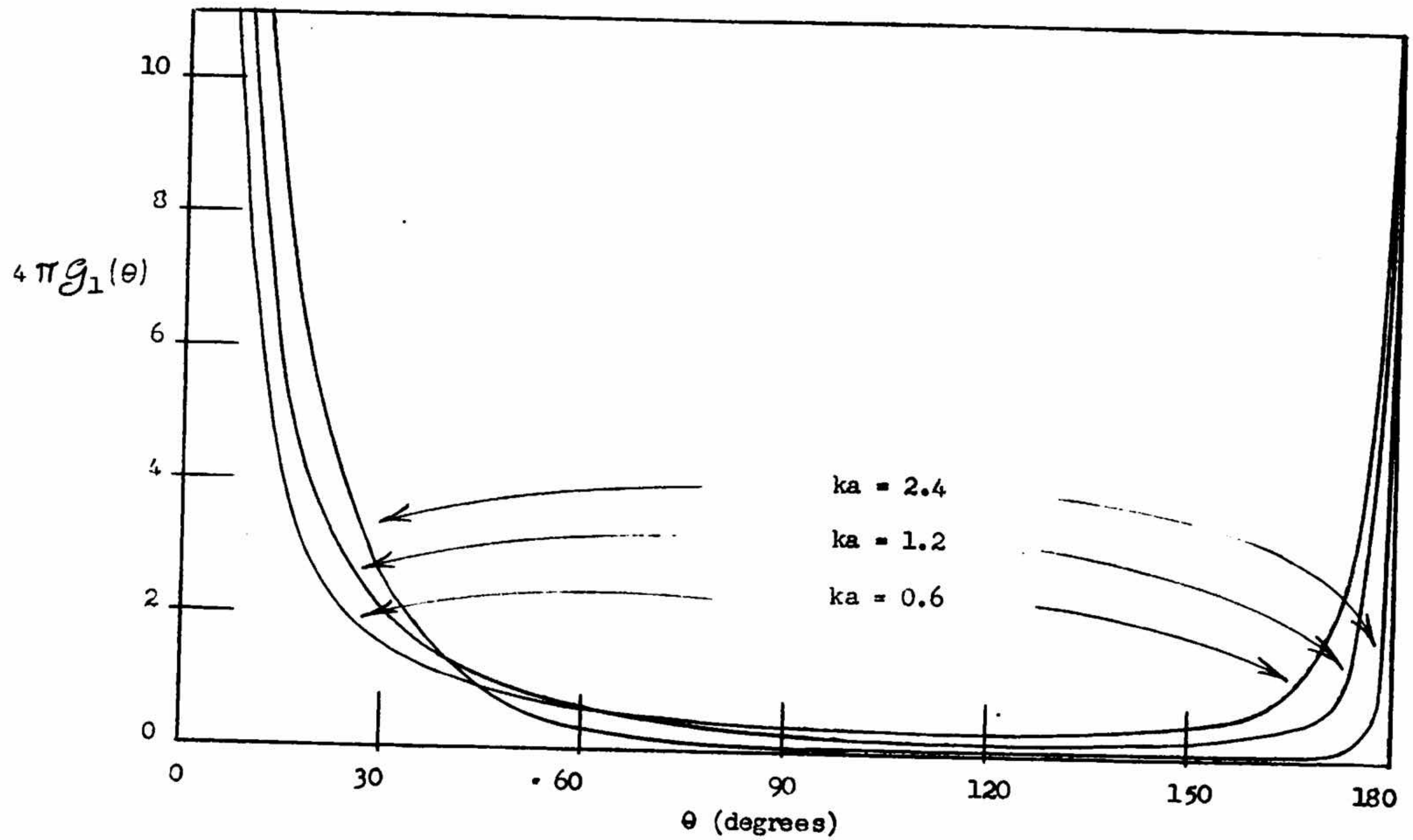
θ in degrees	$k(b-a) = 1.2$, $\frac{h}{a} = 10$	1.2, $10 + \frac{\pi}{1.2}$	1.2, $10 + \frac{\pi}{2.4}$	1.2, $10 - \frac{\pi}{2.4}$
0	∞	∞	∞	∞
2	38.01	552.6	268.7	396.0
8	.3082	48.95	28.47	25.36
15	.5722	16.67	11.64	6.704
30	.8095	4.693	4.342	1.275
45	.6777	1.976	2.199	.4423
60	.4896	.9906	1.229	.2243
75	.3352	.5739	.7421	.1453
90	.2300	.3864	.4919	.10965
105	.1646	.3045	.3642	.09527
120	.1264	.2817	.3047	.09809
135	.1081	.3083	.2919	.12362
150	.1122	.4158	.3351	.1989
165	.1836	.8394	.5613	.4888
172	.3592	1.764	1.0682	1.1275
178	2.145	11.44	6.197	7.971
180	∞	∞	∞	∞



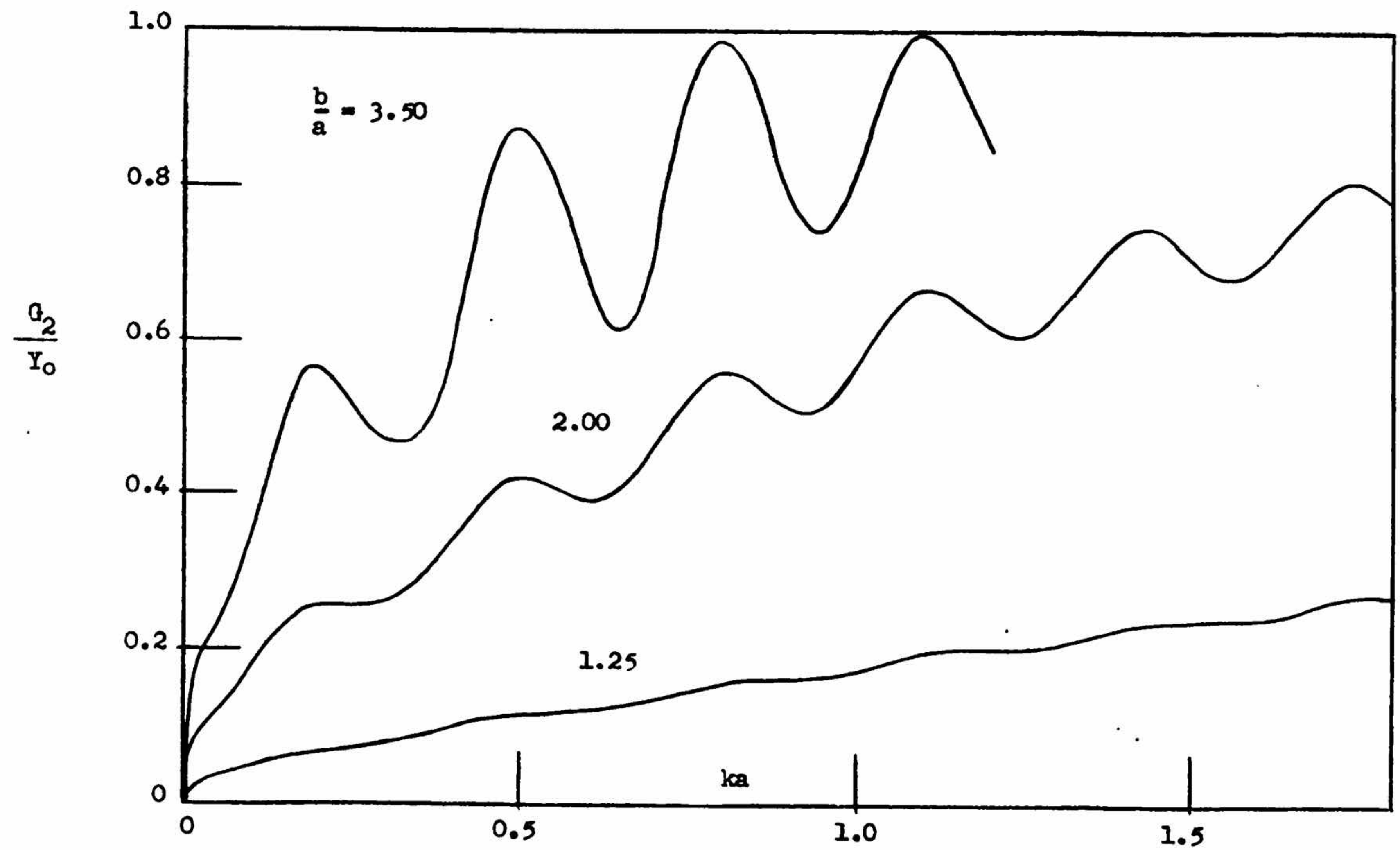
Graph I. Conductance at reference plane, $z = s_1$, as a function of frequency.



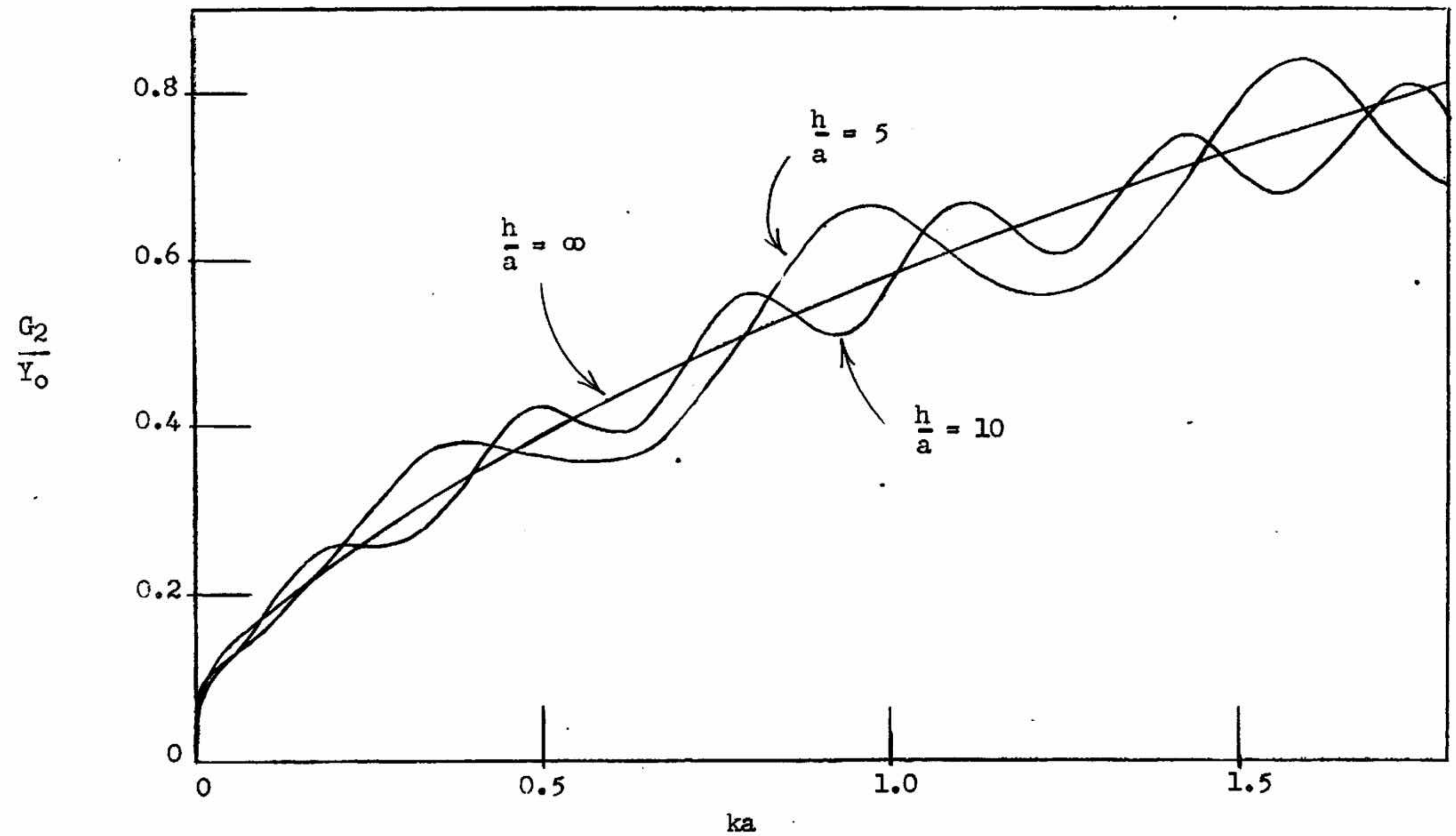
Graph II. Distance s_1 as a function of frequency.



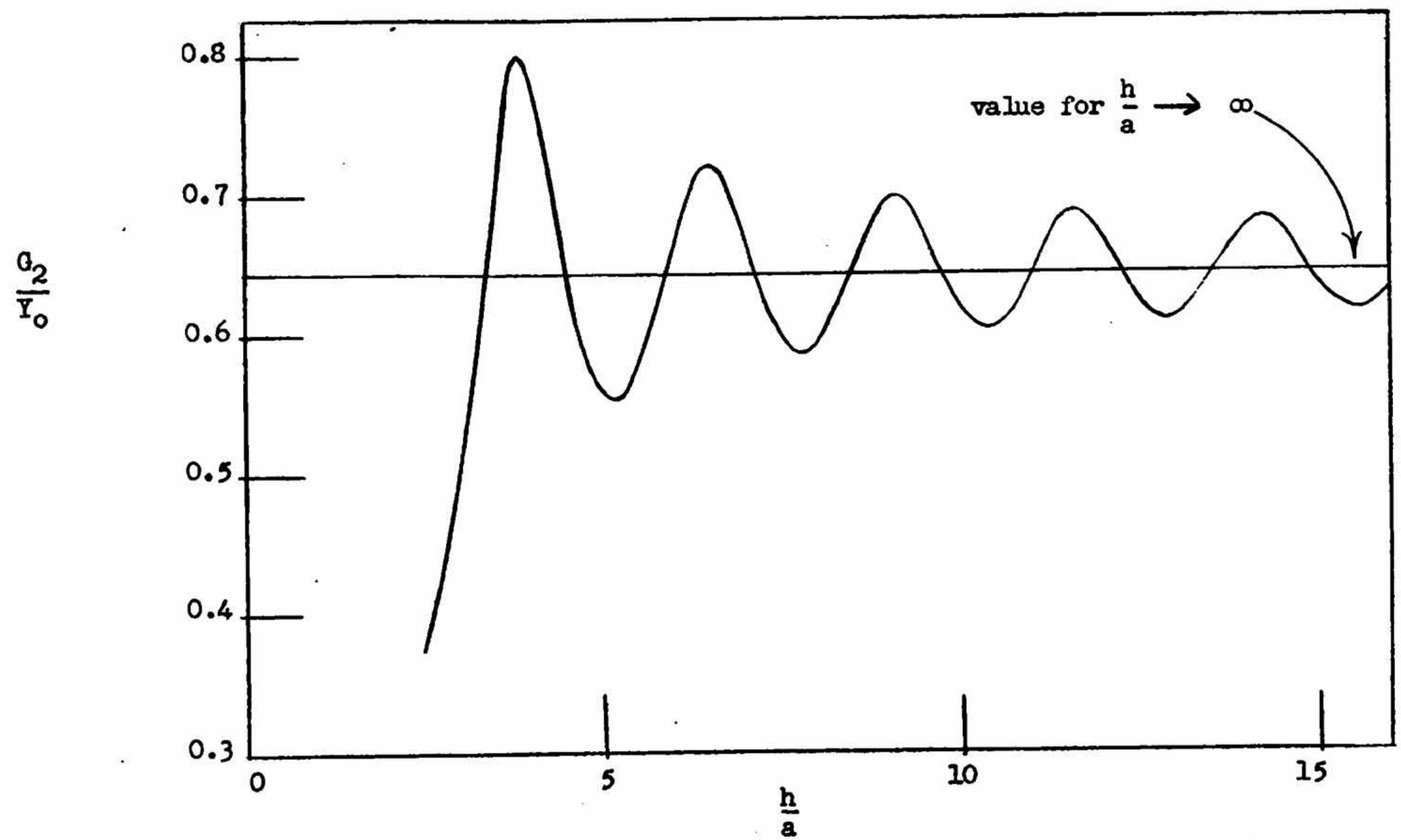
Graph III. Gain function, $\mathcal{G}_1(\theta)$, for $\frac{b}{a} = 2$ as a function of the angle θ.



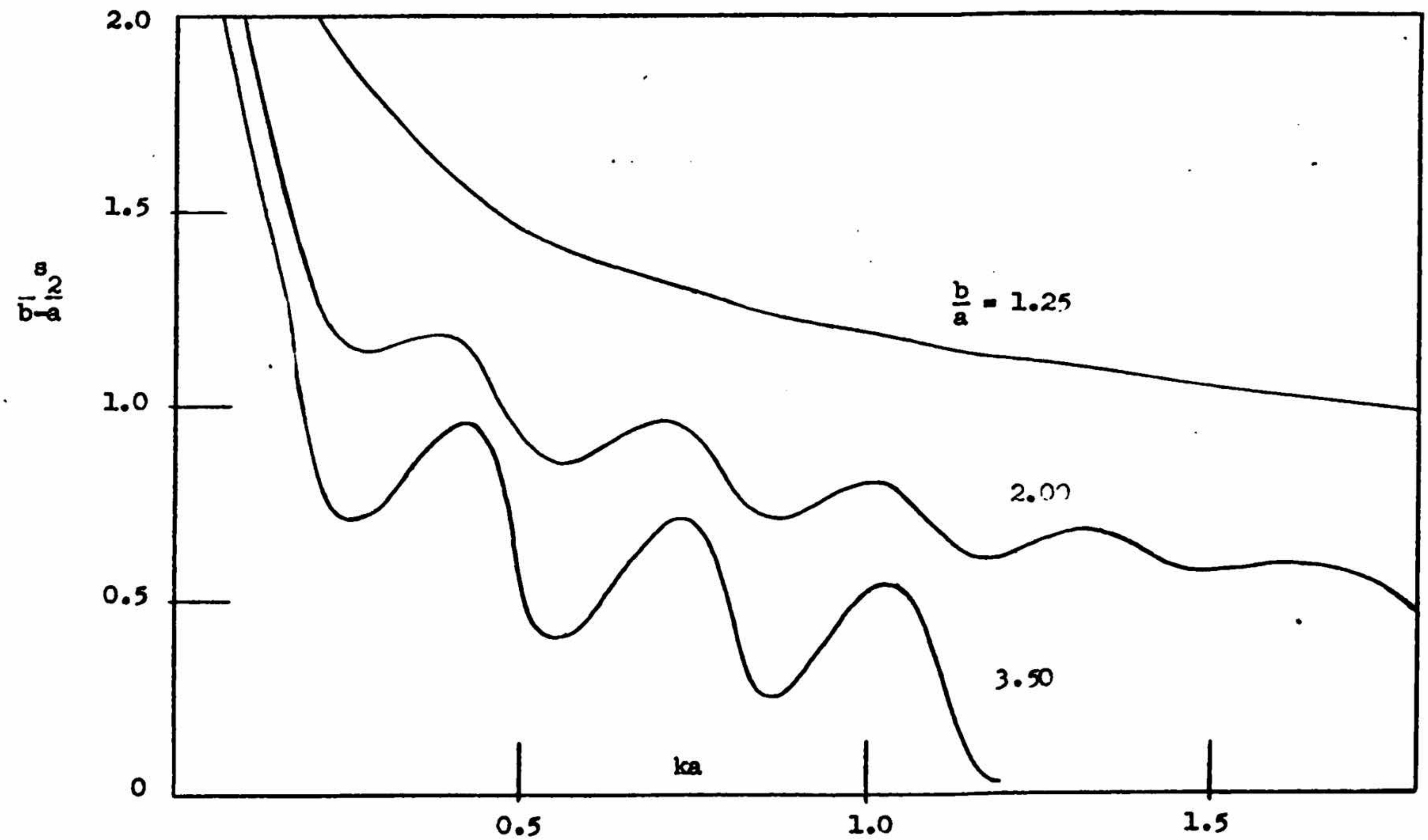
Graph IV. Impedance at the referance plane, $z = s_2$, for $\frac{h}{a} = 10$ as a function of frequency.



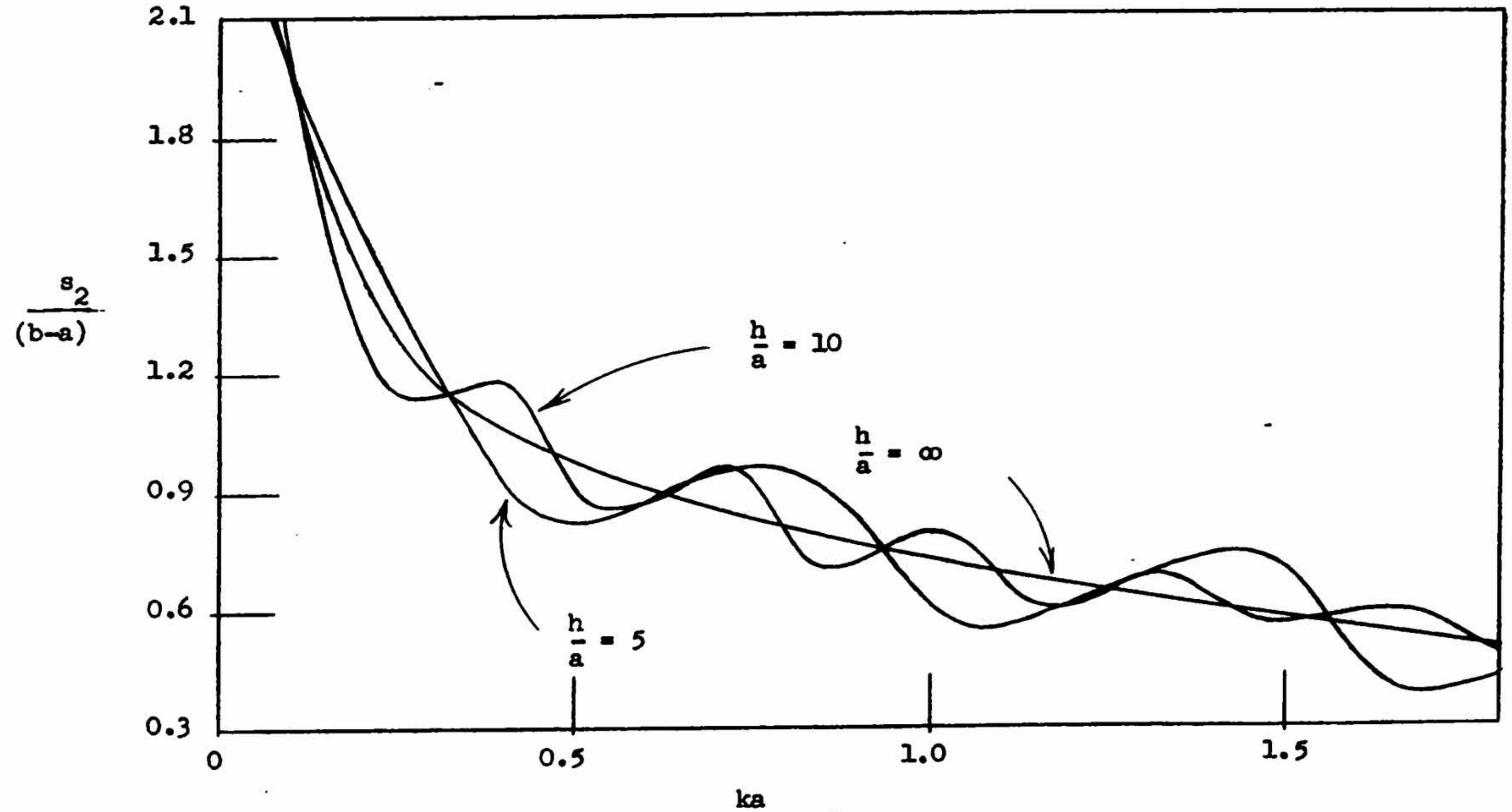
Graph V. Impedance at reference plane, $z = s_2$, for $\frac{b}{a} = 2$ as a function of frequency.



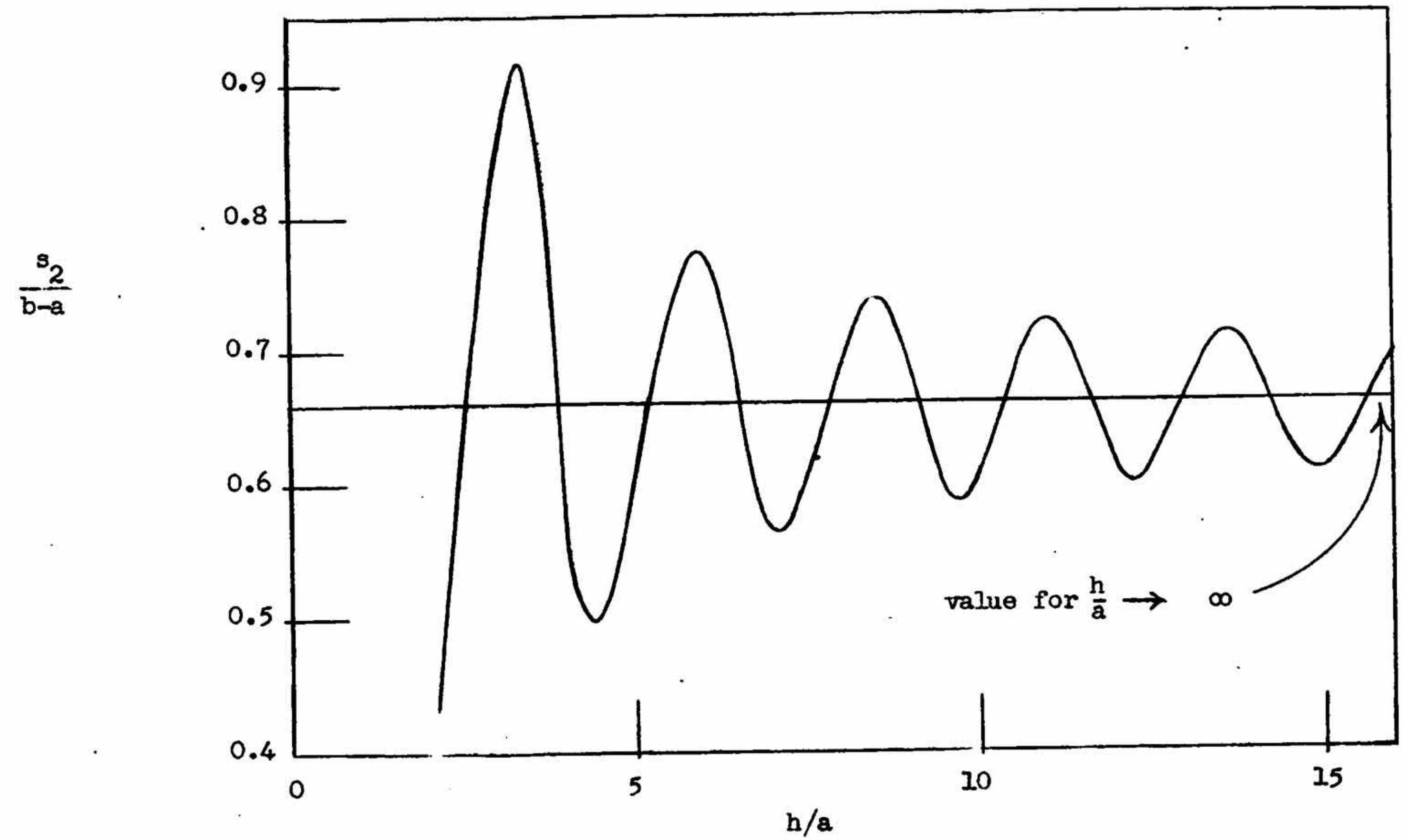
Graph VI. Impedance at reference plane, $z = s_2$, for $ka = 1.2$ and $\frac{b}{a} = 2$ as a function of the ratio, $\frac{h}{a}$.



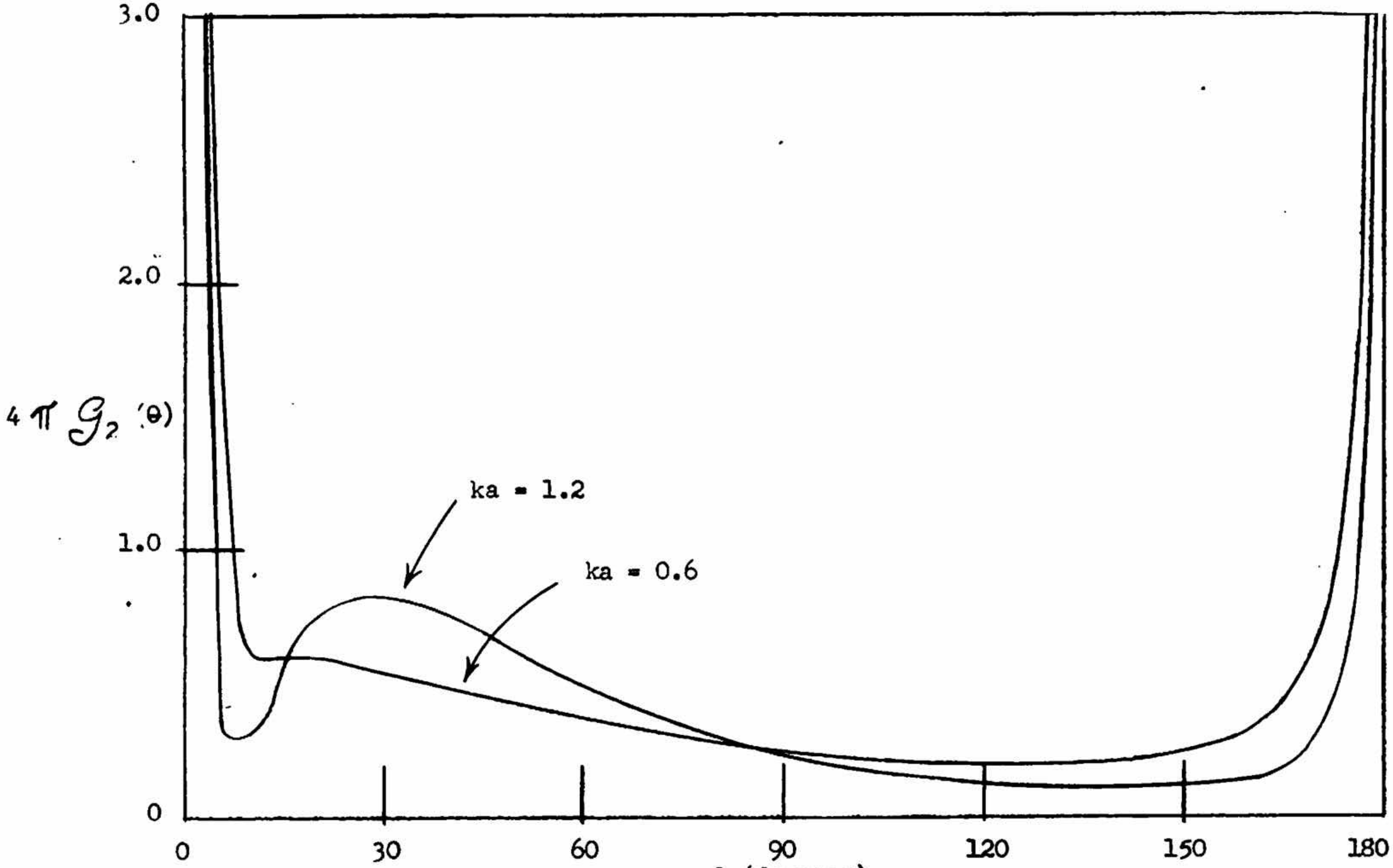
Graph VII. Distance, s_2, for $\frac{h}{a} = 10$ as a function of frequency.



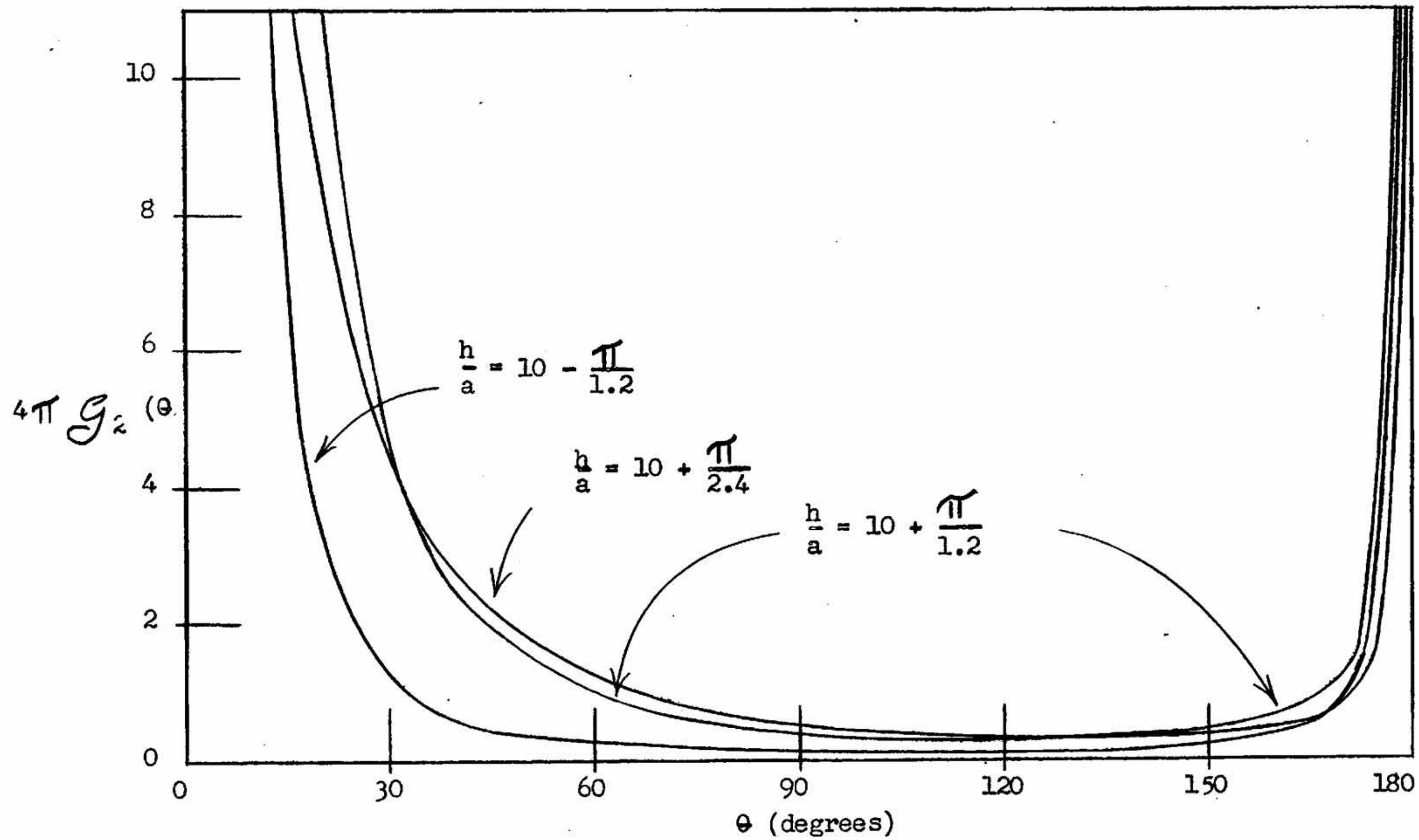
Graph VIII. Distance, s_2, for $\frac{b}{a} = 2$ as a function of frequency.



Graph IX. Distance, s_2, for $ka = 1.2$ and $\frac{b}{a} = 2$, as a function of the ratio, $\frac{h}{a}$.



Graph X. Gain function, $\mathcal{G}_2(\theta)$, for $\frac{h}{a} = 10$ and $\frac{b}{a} = 2$ as a function of the angle, θ.



Graph XI. Gain function, $\mathcal{G}_2$ (θ), for ka = 1.2 and $\frac{b}{a} = 2$ as a function of the angle,0.

Supplementary Table. Values* of S_1 and S_2.

$\frac{b}{a}$	k(b-a)	θ in degrees	$-S_1$	$-S_2$
1.25	0	0	0	
	.30		.0002947	
	1.20		.01237	
	3.00		.3499	
2.00	0		0	0
	.12			.01829
	.15		.0003659	.02288
	.30		.0008661	.04603
	.48			.07458
	.60		.002814	.09432
		15	.002904	.09149
		30	.003060	.08297
		45	.003012	.06882
		60	.002497	.04944
		75	.001432	.02589
		90	0	0
	1.20	0	.01480	.2087
		15	.01583	.2049
		30	.01804	.1920
		45	.01916	.1667
		60	.01682	.1255
		75	.00999	.06803
		90	0	0
	2.40	0	.12934	
		15	.13972	
		30	.16488	
		45	.18715	
		60	.17952	
		75	.11603	
		90	0	
	3.00	0	.3674	
3.50	0		0	
	.30		.002358	
	1.20		.02117	
	3.00		.4174	

*It may be noted that $S_1(\theta) = -S_1(\pi-\theta)$ and $S_2(\theta) = -S_2(\pi-\theta)$.

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Appendix

A. Derivation of the differential equations[1]

For rationalized M.K.S. units and time harmonic solutions in a vacuum (or air) bounded by ideal metals, Maxwell's equations[2] reduce to

$$\nabla \times \vec{E} = ik\zeta \vec{H}, \qquad \nabla \cdot \vec{H} = 0 \qquad (A.1)$$

$$\nabla \times \vec{H} = ik\eta \vec{E}, \qquad \nabla \cdot \vec{E} = 0 \qquad (A.2)$$

subject to the boundary conditions

$$\vec{n} \times \vec{E} = 0$$

$$\vec{n} \cdot \vec{H} = 0$$

where the symbols have their conventional meaning. For a set of differential equations suitable for cylindrical coordinates, consider the following procedure. Taking the curl of (A.2) and substituting the result into (A.1), we obtain

$$\nabla \times \nabla \times \vec{E} - k^2 \vec{E} = 0 .$$

Examining the z component, we have

$$(\nabla^2 + k^2) E_z = 0 . \qquad (A.3)$$

For circular cylindrical coordinates where E_z is axially symmetric, i.e., not a function of the angle ϕ, (A.3) becomes

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[1] D. E. Spencer, Journal of Applied Physics, April, 1951, p. 386, gives a more detailed analysis of the separation of field variables, Ref. (7).

[2] Stratton, op. cit., p. 23, Ref. (1).

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\partial^2}{\partial z^2} + k^2\right) E_z = 0 \; . \qquad (A.4)$$

The transverse components of equation (A.1) may be obtained as follows:

$$\vec{e}_z \times \left\{\left(\nabla_t + \vec{e}_z \frac{\partial}{\partial z}\right) \times (\vec{E}_t + E_z\vec{e}_z)\right\} = ik\zeta\vec{e}_z \times (\vec{H}_t + H_z e_z)$$

which reduces to

$$\frac{\partial \vec{E}_t}{\partial z} = \nabla_t E_z - ik\zeta\vec{e}_z \times \vec{H}_t \qquad (A.5)$$

where the subscript t refers to the transverse components and $\vec{e}_z$ is a unit vector along the z axis. Similarly

$$\frac{\partial \vec{H}_t}{\partial z} = \nabla_t H_z + ik\eta\vec{e}_z \times \vec{E}_t \; . \qquad (A.6)$$

Differentiating (A.5) with respect to z and substituting the value of $\frac{\partial \vec{H}_t}{\partial z}$ given by (A.6), we obtain

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\vec{E}_t = \nabla_t \frac{\partial E_z}{\partial z} + ik\zeta\nabla_t \times (\vec{e}_z H_z) \; . \qquad (A.7)$$

Considering the ρ component and the case of axial symmetry, (A.7) becomes

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) E_\rho = \frac{\partial^2 E_z}{\partial \rho \, \partial z} \quad . \tag{A.8}$$

Similarly

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) \vec{H}_t = - ik\eta \nabla_t X (\vec{e}_z E_z) + \nabla_t \frac{\partial H_z}{\partial z} \quad . \tag{A.9}$$

Considering the ϕ component and the case of axial symmetry, (A.9) becomes

$$\left(\frac{\partial}{\partial z^2} + k^2\right) H_\phi = ik\eta \frac{\partial E_z}{\partial \rho} z \quad . \tag{A.10}$$

Equations (A.4), (A.8), and (A.10) are the desired set of equations. A similar procedure is very useful in the more general problem where sources are included.

The differential equation for H_ϕ,

$$\left(\nabla^2 - \frac{1}{\rho^2} + k^2\right) H_\phi = 0$$

or for the case of axial symmetry

$$\left\{\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho}\right) - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2} + k^2\right\} H_\phi = 0 \ , \tag{A.11}$$

is derived in Stratton[3] by considering the electric type Hertz polarization potential, $\vec{\Pi}$.

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[3]Ibid., pp. 349-50, Ref. (1).

B. Derivation of the Green's function transform, $\mathcal{K}(\zeta)$

The Green's function of interest is a solution of the differential equation

$$\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial z^2}+k^2\right\}K(\rho, \rho', z, z') = -\frac{(\rho-\rho')}{\rho}\delta(z-z'), \quad (B.1)$$

satisfying the boundary condition

$$K(a, \rho', z, z') = 0 \quad (B.2)$$

and the radiation condition for $\vec{r}\to\infty$, $\rho > a$,

$$K \propto \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} . \quad (B.3)$$

Multiplying equation (B.1) by $e^{-i\zeta z}$ and integrating from $z = -\infty$ to $z = +\infty$, we get

$$\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+k^2-\zeta^2\right\}e^{i\zeta z'}\int_{-\infty}^{\infty}K(\rho, \rho', z, z')e^{-i\zeta z}dz$$

$$+\ e^{i\zeta z'}\left[e^{-i\zeta z}\left(\frac{\partial K}{\partial z}+i\zeta K\right)\right]_{z=-\infty}^{z=+\infty} = -\frac{\delta(\rho-\rho')}{\rho} . \quad (B.4)$$

The bracketted team vanishes if k is assumed to have an arbitrarily small imaginary part α and $|\mathrm{Im}\,\zeta| < \alpha$. From the differential equation (B.1) and the symmetry condition, $K(\rho, \rho', z, z') =$

$K(\rho', \rho, z', z)$, we may deduce that

$$K(\rho, \rho', z, z') = K(\rho, \rho', z - z') \; .$$

Then the Green's function transform,

$$\mathcal{K}(\rho, \rho', \zeta) = \int_{-\infty}^{\infty} K(\rho, \rho', z - z')e^{-i\zeta(z-z')}d(z - z') \; , \quad (B.5)$$

is defined for $|\mathrm{Im}\, \zeta| < \alpha$. The differential equation becomes

$$\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + k^2 - \zeta^2\right\}\mathcal{K}(\rho, \rho', \zeta) = -\frac{\delta(\rho - \rho')}{\rho} \; . \quad (B.6)$$

The homogeneous equation, for $\rho \neq \rho'$, may be solved, satisfying the boundary condition $\mathcal{K}(a, \rho', \zeta) = 0$, yielding

$$\mathcal{K}(\rho_>, \rho_>, \zeta) = cH_0^{(1)}(\gamma\rho_>)Z_0(\gamma\rho_<) \qquad (B.7)$$

where

$$Z_0(\gamma\rho_<) = \frac{\pi}{2}\left\{J_0(\gamma a)N_0(\gamma\rho_<) - N_0(\gamma a)J_0(\gamma\rho_<)\right\} \qquad (B.8)$$

where $H_0^{(1)}$ is the zero order Hankel function of the first kind; J_0 and N_0 are the Bessel functions of zero order and of the first and second kind respectively; $\gamma = \sqrt{k^2 - \zeta^2}$; $\rho_>$ refers to whichever is larger ρ or ρ', $\rho_<$ refers to whichever is smaller ρ or ρ'; and c is as yet an undetermined constant.[4] This particular

[4] E. L. Ince, "Ordinary Differential Equations," (Dover, New York) p. 116, Ref. (8).

choice of cylinder functions has been made in order to preserve the sense of an outgoing wave. To find c, equation (B.6) is multiplied by ρ and integrated from $\rho' - 0$ to $\rho' + 0$, to give

$$\rho \frac{\partial \mathcal{K}}{\partial \rho} \Bigg|_{\rho' - 0}^{\rho' + 0} = -1$$

Making use of the values of the Wronskian for two linearly independent solutions to Bessel's equation,[5] we find

$$c = 1/H_0^{(1)}(\gamma a) .$$

Then

$$\mathcal{K}(\rho_>, \rho_<, \zeta) = \frac{H_0^{(1)}(\gamma \rho_>)}{H_0^{(1)}(\gamma a)} Z_0(\gamma \rho_<) . \qquad \text{(B.9)}$$

The general behavior of $\mathcal{K}(b, b, \zeta) \equiv \mathcal{K}(\zeta)$ in the ζ plane is illustrated in Fig.15. The asymptotic behavior of $\mathcal{K}(\zeta)$ for $|\zeta| \to \infty$ is found by substituting the asymptotic forms of the Bessel functions in (B.9) which gives

$$\mathcal{K}(\zeta) \approx -\frac{i}{2\gamma b} \left(e^{2i\gamma(b-a)} - 1\right)$$

Thus $\mathcal{K}(\zeta)$ is bounded in the second and fourth quadrants but becomes infinite in the first and third quadrants for $|\zeta| \to \infty$

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[5] E. Jahnke and F. Emde, "Tables of Functions with Formula and Curves," (Dover Publications, New York, 1945), p. 144, Ref. (9).

and $|\eta| > \alpha$.

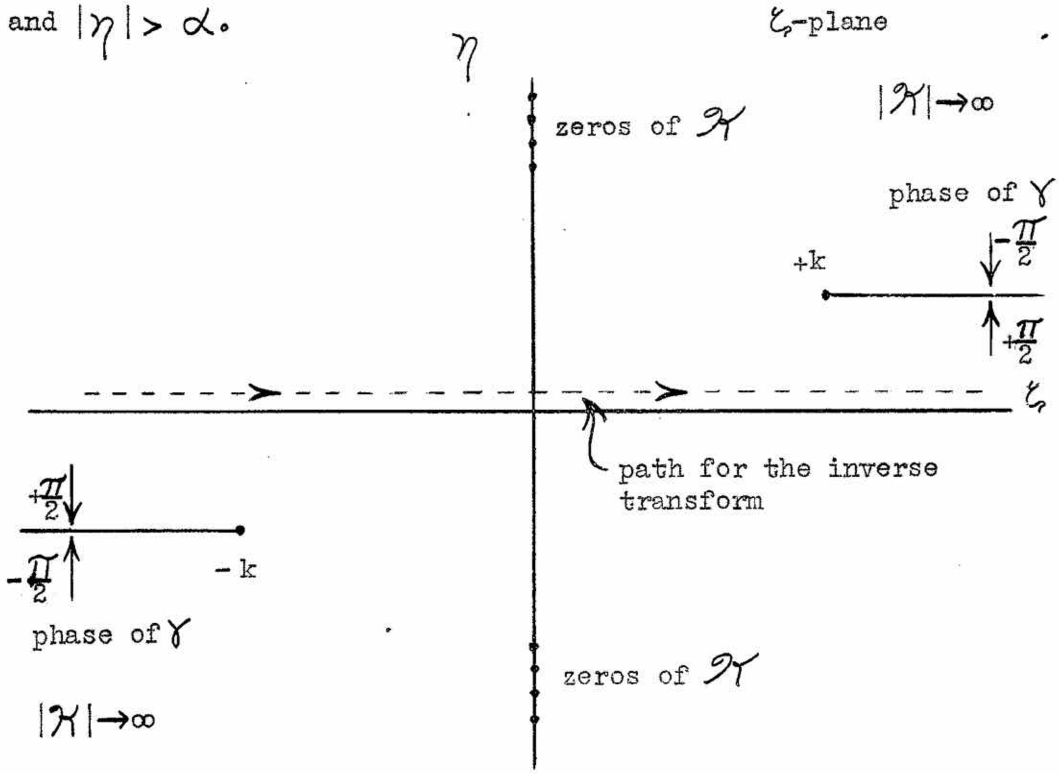

Fig. 15. The behavior of $\mathcal{K}(\zeta)$ in the cut ζ plane.

The value of $\mathcal{K}(\zeta)$ at the branch points is found by substituting the values of the Bessel functions for $\gamma \to 0$ in (B.9),

$$\mathcal{K}(\pm k) = \lim_{\gamma \to 0} \left[\frac{\pi}{2} \frac{1 - \frac{2i}{\pi} \log \frac{2}{\beta\gamma b}}{1 - \frac{2i}{\pi} \log \frac{2}{\beta\gamma a}} \left\{ - \frac{2}{\pi} \log \frac{2}{\beta\gamma b} + \frac{2}{\pi} \log \frac{2}{\beta\gamma a} \right\} \right] = \log \frac{b}{a}$$

where log β = 0.5772157 ... is Euler's constant. It may be shown[6] that $H_o^{(1)}(x)$ has no zeros for $-\pi \leq \arg x \leq \pi$ which is larger than our region of interest; therefore $\mathcal{K}(\zeta)$ has no poles in the finite ζ plane. The factor $Z_o(\gamma b)$ gives $\mathcal{K}(\zeta)$ an infinity of zeros on the imaginary ζ axis for $|\eta| > \alpha$.

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[6]Watson, op. cit., p. 511. Consider $K(x) = \frac{\pi i}{2} H_o^{(1)}(ix)$, Ref. (6).

C. Proof that $\mathcal{G}(\theta)$ is properly normalized

This proof was given by Harold Levine in a private communication to us. Consider the integral

$$\int_c \frac{\mathcal{X}(\zeta)}{|M^+(\zeta)|^2(k^2-\zeta^2)}\, d\zeta \tag{C.1}$$

where the integration contour c is shown in Fig.16 (closed by a circle at infinity).

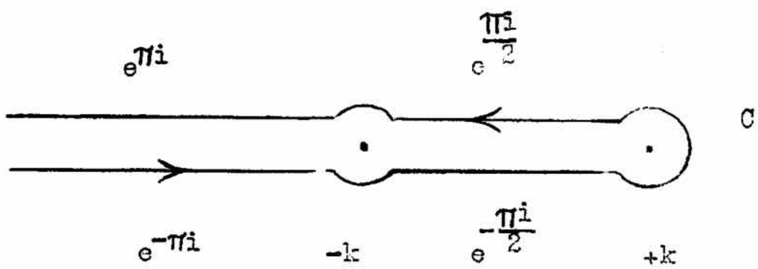

Fig.16. The integration contour for (C.1) showing the phases of $\sqrt{k^2-\zeta^2}$.

Since the contour incloses no singularities, we have, using (3.4),

$$0 = 2\pi i - \frac{\log\frac{b}{a}}{2k|M^+(k)|^2} + 2\pi i\,\frac{\log\frac{b}{a}}{2k|M^+(k)|^2}$$

$$+ P\int_{-k}^{k}\left\{\frac{H_o^{(1)}(\gamma b)}{H_o^{(1)}(\gamma a)} - \frac{H_o^{(1)}(e^{\pi i}\gamma b)}{H_o^{(1)}(e^{\pi i}\gamma a)}\right\}\frac{Z_o(\gamma b)}{|M^+(\zeta)|^2}\,\frac{d\zeta}{(k^2-\zeta^2)}$$

$$+\int_{-\infty\ \text{below cut}}^{-k} + \int_{-k\ \text{above cut}}^{-\infty} \frac{\mathcal{K}(\zeta)}{|M^{+}(\zeta)|^{2}(k^{2}-\zeta^{2})}\, d\zeta \,. \quad (C.2)$$

Substituting in the explicit expressions for $\mathcal{K}(\zeta)$ and $|M^{+}(\zeta)|$ and using the relation

$$H_o^{(1)}\left(e^{\frac{\pi i}{2}}\gamma' a\right) \exp\left[\frac{1}{\pi}\int_0^{\infty} \frac{x \tan^{-1}\left(\frac{K_o(x)}{\pi I_o(x)}\right)}{\sqrt{x^2+k^2a^2}\left[\sqrt{x^2+k^2a^2}+\zeta a - i\epsilon\right]}\, dx\right]$$

$$= - H_o^{(1)}\left(e^{\frac{3\pi i}{2}}\gamma' a\right) \exp\left[\frac{1}{\pi}\int_0^{\infty} \frac{x \tan^{-1}\left(\frac{K_o(x)}{\pi I_o(x)}\right)}{\sqrt{x^2+k^2a^2}\left[\sqrt{x^2+k^2a^2}+\zeta a + i\epsilon\right]}\, dx\right]$$

for $\epsilon \rightarrow 0$, and similarly for b, it may be shown that the last two integrals in (C.2) cancel, i.e., $\mathcal{K}(\zeta)/|M^{+}(\zeta)|^{2}$ does not change across the cut. Changing the variable of integration, $\zeta = k \cos\theta$, equation (C.2) becomes

$$0 = \frac{\pi i}{k}\left(-\frac{1}{|R_1|} + |R_1|\right)$$

$$+ \frac{i}{k}\int_0^{\pi} \frac{4\, Z_o^{2}(kb \sin\theta) \sin\theta}{\pi |H_o^{(1)}(ka \sin\theta)|^{2} |M^{+}(k\cos\theta)|^{2} \sin^{2}\theta}$$

which checks with (9.18) when (9.17) is used for $g_1(\theta)$.

D. Derivation of the free space Green's function transform, $\mathcal{G}(\rho_>, \rho_<, \zeta)$

The free space Green's function is

$$G(\vec{r}, \vec{r}') = \frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \qquad (D.1)$$

which satisfies

$$(\nabla^2 + k^2)G(\vec{r}, \vec{r}') = -\delta(\vec{r} - \vec{r}') \qquad (D.2)$$

For the present case of axial symmetry (D.2) becomes

$$\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\partial^2}{\partial z^2} + k^2\right\} G(\rho, \rho', z - z')$$

$$= -\frac{\delta(z - z')\delta(\rho - \rho')}{\rho} . \qquad (D.3)$$

Multiplying (D.3) by $e^{-i\zeta z}$ and integrating with respect to z from $-\infty$ to $+\infty$, we get

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + k^2\right]\int_{-\infty}^{\infty} Ge^{-i\zeta z}\, dz \qquad (D.4)$$

$$+ \int_{-\infty}^{\infty} e^{-i\zeta z}\frac{\partial^2 G}{\partial z^2}\, dz = -\frac{\delta(\rho - \rho')}{\rho}\int_{-\infty}^{\infty}\delta(z - z')e^{-i\zeta z}\, dz .$$

The second integral on the left of (D.4) may be integrated by parts to yield

$$\left[\left(\frac{\partial G}{\partial z} + i \zeta G \right) e^{-i\zeta z} \right]_{z=-\infty}^{z=+\infty} - \zeta^2 \int_{-\infty}^{\infty} G e^{-i\zeta z} \, dz$$

The bracketted term is zero provided $|\mathrm{Im}\,\zeta| < \alpha$ where $k = \beta + i\alpha$. Defining the Fourier transform

$$\mathcal{G}(\rho, \rho', \zeta) = \int_{-\infty}^{\infty} G(\rho, \rho', z - z') e^{-i\zeta z} \, dz$$

$$= \int_{-\infty}^{\infty} G(\rho, \rho', z - z') e^{-i\zeta(z-z')} d(z - z') ,$$

equation (D.4) becomes

$$\left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \gamma^2 \right\} \mathcal{G}(\rho, \rho', \zeta) = - \frac{\delta(\rho - \rho')}{\rho} \qquad (D.5)$$

where $\gamma = \sqrt{k^2 - \zeta^2}$. Solving the homogeneous equation,

$$\mathcal{G}(\rho, \rho', \zeta) = c J_0(\gamma \rho_<) H_0^{(1)}(\gamma \rho_>) \qquad (D.6)$$

where the Bessel functions have been chosen to preserve the sense of an outgoing wave. Multiplying (D.5) by ρ and integrating from $\rho' - 0$ to $\rho' + 0$, we get

$$\rho \frac{\partial}{\partial \rho} \mathcal{G}(\rho, \rho', \zeta) \Big|_{\rho = \rho' - 0}^{\rho = \rho' + 0} = -1 . \qquad (D.7)$$

Using the value of the Wronskian of two linearly independent solutions of Bessel's equation[7], we find $c = \frac{\pi i}{2}$. Thus

$$\mathcal{G}(\rho_>, \rho_<, \zeta) = \frac{\pi i}{2} J_0(\gamma \rho_<) H_0^{(1)}(\gamma \rho_>) . \qquad (D.8)$$

Fig.15 in Appendix A shows the cut ζ-plane and how the phase of γ is chosen.

In particular we are interested in $\mathcal{G}(a, a, \zeta) \equiv \mathcal{G}(\zeta)$. The asymptotic behavior is found by substituting in the asymptotic form of the Bessel Functions.

$$\mathcal{G}(\zeta) \approx \frac{i}{2a\gamma}\left(e^{2i\gamma - \frac{\pi i}{2}} + 1\right) \text{for } |\zeta| \to \infty . \qquad (D.9)$$

For ζ approaching the branch points we have

$$\mathcal{G}(\zeta) \approx \frac{\pi i}{2} + \log \frac{2}{\beta a \gamma} \quad \text{for } |\zeta| \to k \qquad (D.10)$$

where $\log \beta = 0.5772 \cdots$ is Euler's constant, obtained by considering the Bessel functions for a small argument. Thus $\mathcal{G}(\zeta)$ has a logarithmic singularity at the branch points. $\mathcal{G}(\zeta)$ has zeros at $\zeta_n = \pm i\sqrt{\gamma_n^2 - k^2}$ where $J_0(\gamma_n a) = 0$.